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F-contraction on asymmetric metric spaces

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Abstract

In this paper, we introduce the notion of an F-contraction in the setting of complete asymmetric metric spaces and we investigate the existence of fixed points of such mappings. Our results unify, extend, and improve several results in the literature.

Keywords: Fixed point, asymmetric metric spaces, F-contraction.
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1. Introduction

In 1931, for the first time asymmetric metric spaces were introduced by Wilson [16] as quasi-metric spaces, and then studied by many authors (see [1, 6, 8, 11]). An asymmetric metric space is a generalization of a metric space, but without the requirement that the (asymmetric) metric ρ has to satisfy $\rho(x, y) = \rho(y, x)$. In asymmetric metric spaces some notions, such as convergence, compactness and completeness are different from this in metric case. There are two notions for each of them, namely forward and backward ones, since we have two topologies which are the forward topology and the backward topology in asymmetric metric spaces (see [5]). Collins and Zimmer [2] studied these notions in the asymmetric context. Asymmetric metrics have many applications in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton-Jacobi equations [8] in mind.

In recent years an interesting but different generalizations of the Banach-contraction theorem have been given by Wardowski [14]. This result have become of recent interest of many authors (see [3, 4, 10, 12, 13, 15] and references therein).

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Most recently, Piri and Kumam [9], proved a fixed point result for F-contractions for some weaker conditions on the self-map of a complete metric space which generalizes the result of Wardowski.

By following this direction of research, in this paper, we present some new fixed point results for F-contraction on complete asymmetric metric spaces. Moreover, an example is considered to illustrate the usability of the obtained results. Our result is generalization of the results announced by Wardowski [14] and Piri and Kumam [9].

2. Preliminaries

The aim of this section is to present some notions and results used in the paper. Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively.

Definition 2.1. Let $\mathcal{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(F1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$, if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.2 ([14])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, d), if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y))].$$

Wardowski [14] stated a modified version of Banach contraction principle as follows:

Theorem 2.3 ([14])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Very recently, Piri and Kumam [9] described a large class of functions by replacing the condition (F3) in the definition of F-contraction introduced by Wardowski [14] with the following one:

(F3′) F is continuous on $(0, \infty)$.

Piri and Kumam used $\mathfrak{F}$ to denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F1), (F2) and (F3′). Under this new set-up, they proved Wardowski type fixed point results in metric spaces as follows:

Theorem 2.4 ([9])**.** *Let T be a self-mapping of a complete metric space X into itself. Suppose, there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y))].$$

Then, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.

Now, we recall some definitions and results on asymmetric metric spaces.

Definition 2.5 ([2])**.** A mapping $\rho : X \times X \to \mathbb{R}_0^+$, where X is a nonempty set, is said to be an asymmetric metric on X, if for any $x, y, z \in X$, the following conditions hold:

(am1) $\rho(x, y) = 0, \text{ if and only if } x = y;$

(am2) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$.

The pair (X, ρ) is called an asymmetric metric space.

Definition 2.6 ([2])**.** Let (X, ρ) be an asymmetric metric space.

(i) The forward topology τ_+ induced by ρ is the topology generated by the forward open balls

$$B_+(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\}, \quad \forall x \in X, \quad \epsilon > 0.$$

(ii) The backward topology τ_- induced by ρ is the topology generated by the backward open balls

$$B_-(x, \epsilon) = \{y \in X : \rho(y, x) < \epsilon\}, \quad \forall x \in X, \quad \epsilon > 0.$$

Example 2.7 ([7])**.** The function $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$\rho(x, y) = \begin{cases} e^y - e^x, & \text{if } y \geqslant x, \\ e^{-y} - e^{-x}, & \text{if } y < x, \end{cases}$$

is an asymmetric metric. Both τ_+ and τ_- are the usual topologies on $\mathbb{R}$.

Definition 2.8 ([2])**.** Let (X, ρ) be an asymmetric metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$. Then

(i) We say that $\{x_n\}_{n=1}^{\infty}$ forward converges to x, if and only if

$$\lim_{n\to\infty} d(x, x_n) = 0.$$

(ii) We say that $\{x_n\}_{n=1}^{\infty}$ backward converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

(iii) We say that $\{x_n\}_{n=1}^{\infty}$ forward Cauchy, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m > n > N$, $d(x_n, x_m) < \epsilon$ holds, or equivalently

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

(v) We say that $\{x_n\}_{n=1}^{\infty}$ backward Cauchy, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m > n > N$, $d(x_m, x_n) < \epsilon$ holds, or equivalently

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Lemma 2.9 ([2])**.** *Let (X, ρ) be an asymmetric metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Definition 2.10 ([2])**.** Let (X, ρ) be an asymmetric metric space.

(i) X is said to be forward complete, if every forward Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X forward converges to $x \in X$.

(ii) X is said to be backward complete, if every backward Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X backward converges to $x \in X$.

Definition 2.11. Let (X, ρ) be an asymmetric metric space. X is said to be complete, if X is forward and backward complete.

3. Main results

Let $\mathfrak{F}_G$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F1) and (F3′) and $\mathcal{F}_G$ denotes the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions (F1) and (F3).

Definition 3.1. Let (X, ρ) be an asymmetric metric space and $T : X \to X$ be a mapping. T is said to be an F-contraction of type (A), if there exist $F \in \mathfrak{F}_G$ and $\tau > 0$ such that for all $x, y \in X$

$$\max\{\rho(Tx,Ty),\rho(Ty,Tx)\} > 0 \Rightarrow \tau + F\left(\frac{\rho(Tx,Ty)+\rho(Ty,Tx)}{2}\right) \leqslant F\left(\frac{\rho(x,y)+\rho(y,x)}{2}\right).$$

Definition 3.2. Let (X, ρ) be an asymmetric metric space and $T : X \to X$ be a mapping. T is said to be an F-contraction of type (B), if there exist $F \in \mathcal{F}_G$ and $\tau > 0$ such that for all $x, y \in X$

$$\max\{\rho(Tx,Ty),\rho(Ty,Tx)\} > 0 \Rightarrow \tau + F\left(\frac{\rho(Tx,Ty)+\rho(Ty,Tx)}{2}\right) \leqslant F\left(\frac{\rho(x,y)+\rho(y,x)}{2}\right).$$

Theorem 3.3. *Let (X, ρ) be a complete asymmetric metric space and let $T\colon X \to X$ be an F-contraction of type (A). Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ a sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Choose $x \in X$. Set $x_{n+1} = Tx_n = T^{n+1}x$, for all $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}$ such that $\rho(x_n, Tx_n) = 0$ or $\rho(Tx_n, x_n) = 0$, then the proof is complete. So, we assume that

$$\rho(x_n, Tx_n) > 0 \text{ and } \rho(Tx_n, x_n) > 0, \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\max\{\rho(Tx_{n-1}, Tx_n), \rho(Tx_n, Tx_{n-1})\} > 0, \quad \forall n \in \mathbb{N}.$$

So from assumption of the theorem, we get,

$$\tau + F\left(\frac{\rho(Tx_{n-1},Tx_n)+\rho(Tx_n,Tx_{n-1})}{2}\right) \leqslant F\left(\frac{\rho(x_{n-1},x_n)+\rho(x_n,x_{n-1})}{2}\right), \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Now, we set

$$D(x_n, x_m) = \rho(x_n, x_m) + \rho(x_m, x_n), \quad \forall m, n \in \mathbb{N}.$$

It follows from (3.1) and (F1) that $D(x_n, x_{n+1}) \leqslant D(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. Therefore $\{D(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus, there exists $\delta \geqslant 0$ such that

$$\lim_{n\to\infty} D(x_{n+1}, x_n) = \delta.$$

Now, we claim that $\delta = 0$. Arguing by contradiction, we assume that $\delta > 0$. Since $\{D(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, so we have

$$D(x_n, x_{n+1}) \geqslant \delta, \qquad \forall n \in \mathbb{N}.$$

From (3.1) and (F1), we get

$$\begin{aligned} F(\frac{\delta}{2}) \leqslant F\left(\frac{D(x_n,x_{n+1})}{2}\right) &\leqslant F\left(\frac{D(x_{n-1},x_n)}{2}\right) - \tau \\ &\leqslant F\left(\frac{D(x_{n-2},x_{n-1})}{2}\right) - 2\tau \\ &\vdots \\ &\leqslant F\left(\frac{D(x_0,x_1)}{2}\right) - n\tau, \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. Since $F(\frac{\delta}{2}) \in \mathbb{R}$ and $\lim_{n\to\infty}[F(D(x_0, x_1)) - n\tau] = -\infty$. So there exists $n_1 \in \mathbb{N}$ such that

$$F((d(x_0, x_1)) - n\tau < F(\frac{\delta}{2}), \quad \forall n > n_1. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$F(\frac{\delta}{2}) \leqslant F(d(x_0, x_1)) - n\tau < F(\frac{\delta}{2}), \quad \forall n > n_1.$$

It is a contradiction. Therefore, we have

$$\lim_{n\to\infty} D(x_n, Tx_n) = \lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{3.4}$$

Now, we claim that, $\lim_{n,m\to\infty} D(x_n, x_m) = 0$. Suppose to the contrary that there exists $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \quad D(x_{p(n)}, x_{q(n)}) \geqslant \epsilon, \quad D(x_{p(n)-1}, x_{q(n)}) < \epsilon.$$

So, we have

$$\begin{aligned}
\epsilon &\leqslant D(x_{p(n)}, x_{q(n)}) \\
&= \rho(x_{p(n)}, x_{q(n)}) + \rho(x_{q(n)}, x_{p(n)}) \\
&\leqslant \rho(x_{p(n)}, x_{p(n)-1}) + \rho(x_{p(n)-1}, x_{q(n)}) + \rho(x_{q(n)}, x_{p(n)-1}) + \rho(x_{p(n)-1}, x_{q(n)}) \\
&= D(x_{p(n)}, x_{p(n)-1}) + D(x_{p(n)-1}, x_{q(n)}) \\
&\leqslant D(x_{p(n)}, x_{p(n)-1}) + \epsilon.
\end{aligned}$$

It follows from (3.4) and Sandwich Theorem that

$$\lim_{n\to\infty} D(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{3.5}$$

Again by the triangle inequality, for all $n \in \mathbb{N}$, we have the following two inequalities:

$$\begin{aligned}
D(x_{p(n)+1}, x_{q(n)+1}) &= \rho(x_{p(n)+1}, x_{q(n)+1}) + \rho(x_{q(n)+1}, x_{p(n)+1}) \\
&\leqslant \rho(x_{p(n)+1}, x_{p(n)}) + \rho(x_{p(n)}, x_{q(n)}) + \rho(x_{q(n)}, x_{q(n)+1}) \\
&\quad + \rho(x_{q(n)+1}, x_{q(n)}) + \rho(x_{q(n)}, x_{p(n)}) + \rho(x_{p(n)}, x_{p(n)+1}) \\
&= D(x_{p(n)+1}, x_{p(n)}) + D(x_{p(n)}, x_{q(n)}) + D(x_{q(n)}, x_{q(n)+1}),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
D(x_{p(n)}, x_{q(n)}) &= \rho(x_{p(n)}, x_{q(n)}) + \rho(x_{q(n)}, x_{p(n)}) \\
&\leqslant \rho(x_{p(n)}, x_{p(n)+1}) + \rho(x_{p(n)+1}, x_{q(n)+1}) + \rho(x_{q(n)+1}, x_{q(n)}) \\
&\quad + \rho(x_{q(n)}, x_{q(n)+1}) + \rho(x_{q(n)+1}, x_{p(n)+1}) + \rho(x_{p(n)+1}, x_{p(n)}) \\
&= D(x_{p(n)}, x_{p(n)+1}) + D(x_{p(n)+1}, x_{q(n)+1}) + D(x_{q(n)+1}, x_{q(n)}).
\end{aligned} \tag{3.7}$$

By letting $n \to \infty$ in the inequalities (3.6) and (3.7) and using (3.4) and (3.5), we obtain

$$\lim_{n\to\infty} [\rho(Tx_{p(n)}, Tx_{q(n)}) + \rho(Tx_{q(n)}, Tx_{p(n)})] = \lim_{n\to\infty} D(x_{p(n)+1}, x_{q(n)+1}) = \epsilon.$$

So, there exists $n_2 \in \mathbb{N}$ such that

$$\rho(Tx_{p(n)}, Tx_{q(n)}) + \rho(Tx_{q(n)}, Tx_{p(n)}) \geqslant \frac{\epsilon}{2}, \quad \text{for all } n \geqslant n_2.$$

Therefore
$$\max\{\rho(Tx_{p(n)}, Tx_{q(n)}), \rho(Tx_{q(n)}, Tx_{p(n)})\} \geqslant \frac{\epsilon}{4}, \quad \forall n \geqslant n_2.$$
So from assumption of the theorem, we get
$$\tau + F\left(\frac{D(x_{p(n)+1}, x_{q(n)+1})}{2}\right) \leqslant F\left(\frac{D(x_{p(n)}, x_{q(n)})}{2}\right).$$
It follows from (3.5) and (F3′) that $\tau + F(\frac{\epsilon}{2}) \leqslant F(\frac{\epsilon}{2})$, which is a contradiction, it follows that
$$\lim_{n,m\to\infty} D(x_n, x_m) = 0$$
and therefore
$$\lim_{n,m\to\infty} \rho(x_m, x_n) = 0, \quad \text{and} \quad \lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$
Hence $\{x_n\}$ is forward and backward Cauchy sequence in X. By completeness of (X, d), there exist x^*, y^* in X such that
$$\lim_{n\to\infty} \rho(x_n, x^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} \rho(y^*, x_n) = 0.$$
So, from Lemma 2.9, we get $x^* = y^*$ and hence
$$\lim_{n\to\infty} \rho(x_n, x^*) = \lim_{n\to\infty} \rho(x^*, x_n) = 0. \tag{3.8}$$

In this stage, we show that, $\rho(x^*, Tx^*) = 0$ and $\rho(Tx^*, x^*) = 0$. Arguing by contradiction, we assume that $\rho(x^*, Tx^*) > 0$ or $\rho(Tx^*, x^*) > 0$. First assume that $\rho(x^*, Tx^*) > 0$. Observe that
$$\begin{aligned}\rho(x^*, Tx^*) &\leqslant \rho(x^*, x_{n+1}) + \rho(x_{n+1}, Tx^*)\\ &\leqslant \rho(x^*, x_{n+1}) + \rho(x_{n+1}, x^*) + \rho(x^*, Tx^*).\end{aligned} \tag{3.9}$$
It follows from (3.8) and (3.9) that
$$\lim_{n\to\infty} \rho(Tx_n, Tx^*) = \rho(x^*, Tx^*). \tag{3.10}$$
So there exists $n_3 \in \mathbb{N}$ such that
$$\max\{\rho(Tx_n, Tx^*), \rho(Tx^*, Tx_n)\} \geqslant (Tx_n, Tx^*) > \frac{\rho(x^*, Tx^*)}{2} > 0, \quad \text{for all } n \geqslant n_3.$$
So, from assumption of theorem, for all $n \geqslant n_3$, we have
$$\tau + F\left(\frac{\rho(Tx_n, Tx^*) + \rho(Tx^*, Tx_n)}{2}\right) \leqslant F\left(\frac{\rho(x_n, x^*) + \rho(x^*, x_n)}{2}\right),$$
and from (F1), we obtain
$$0 \leqslant \rho(Tx_n, Tx^*) + \rho(Tx^*, Tx_n) \leqslant \rho(x_n, x^*) + \rho(x^*, x_n), \quad \forall n \in \mathbb{N}.$$
It follows from (3.8) and Sandwich Theorem that
$$\lim_{n\to\infty} \rho(Tx_n, Tx^*) + \rho(Tx^*, Tx_n) = 0,$$

and hence

$$\lim_{n\to\infty} \rho(Tx_n, Tx^*) = \lim_{n\to\infty} \rho(Tx^*, Tx_n) = 0.$$

So from (3.10), we get $\rho(x^*, Tx^*) = 0$ and this is contradiction. If $\rho(Tx^*, x^*) > 0$, by using similar method, we get contradiction. Therefore $\rho(x^*, Tx^*) = 0$ and $\rho(Tx^*, x^*) = 0$, hence $x^* = Tx^*$. Now, suppose that y^* is another fixed point of T such that $\rho(x^*, y^*) > 0$. Therefore

$$\max\{\rho(Tx^*, y^*), \rho(x^*, Ty^*)\} > 0.$$

So from assumption of theorem, we get

$$\begin{aligned}\tau + F\left(\frac{\rho(x^*, y^*) + \rho(y^*, x^*)}{2}\right) &= F\left(\frac{\rho(Tx^*, Ty^*) + \rho(Ty^*, Tx^*)}{2}\right)\\ &\leqslant F\left(\frac{\rho(x^*, y^*) + \rho(y^*, x^*)}{2}\right),\end{aligned}$$

which is a contradiction. Then T has only one fixed point. □

Theorem 3.4. *Let (X, ρ) be a complete asymmetric metric space and let $T\colon X \to X$ be an F-contraction of type (B). Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ a sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Choose $x \in X$. Set $x_{n+1} = Tx_n = T^{n+1}x$, for all $n \in \mathbb{N}_0$ and $D(x_n, x_m) = \rho(x_n, x_m) + \rho(x_m, x_n)$, for all $m, n \in \mathbb{N}$. As used in the proof of Theorem 3.3 , we can prove that

$$\lim_{n\to\infty} D(x_n, Tx_n) = \lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{3.11}$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\frac{D(x_n, x_{n+1})}{2}\right)^k F\left(\frac{D(x_n, x_{n+1})}{2}\right) = 0. \tag{3.12}$$

On the other hand from assumption of theorem, we get

$$F\left(\frac{D(x_n, x_{n+1})}{2}\right) \leqslant F\left(\frac{D(x_0, x_1)}{2}\right) - n\tau, \quad \forall n \in \mathbb{N}.$$

It follows that for all $n \in \mathbb{N}$

$$\begin{aligned}\left(\frac{D(x_n, x_{n+1})}{2}\right)^k F\left(\frac{D(x_n, x_{n+1})}{2}\right) - \left(\frac{D(x_n, x_{n+1})}{2}\right)^k F\left(\frac{D(x_0, x_1)}{2}\right) &\leqslant -n\tau \left(\frac{D(x_n, x_{n+1})}{2}\right)^k\\ &\leqslant 0.\end{aligned}$$

By letting $n \to \infty$ in the above inequality and using (3.11) and (3.12), we obtain

$$\lim_{n\to\infty} n \left(\frac{D(x_n, x_{n+1})}{2}\right)^k = 0.$$

So, there exists $N \in \mathbb{N}$ such that

$$D(x_n, x_{n+1}) \leqslant \frac{2}{n^{\frac{1}{k}}}, \quad \forall n \geqslant N.$$

Hence for all $m, n \in \mathbb{N}$ such that $n > m \geqslant N$, we obtain

$$D(x_n, x_m) \leqslant \sum_{i=m}^{\infty} \frac{2}{i^{\frac{1}{k}}}.$$

From the above and from the convergence of the series $\sum_{i=m}^{\infty} \frac{2}{i^{\frac{1}{k}}}$, we get

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

As used in the proof of Theorem 3.3, we can prove that $\{x_n\}$ converges to $x^* \in X$ and $x^* = Tx^*$. □

Remark 3.5. Our theorems are extensions of Theorem 2.1 of [9] and Theorem 2.1 of [14] in the following aspects.

(1) Theorem 3.3 is generalization of Theorem 2.1 of [9] from metric space to asymmetric metric space.

(2) Theorem 3.3 gives all consequence of Theorem 2.1 of [9] without assumptions (F2) used in its proof.

(3) Theorem 3.4 is generalization of Theorem 2.1 of [14] from metric space to asymmetric metric space.

(4) Theorem 3.4 gives all consequence of Theorem 2.1 of [14] without assumption (F2) used in its proof.

Example 3.6. Consider $X = \{0, 1, 2\}$. Let $\rho : X \times X \to [0, \infty)$ be a mapping defined by

$$\rho(0,0) = \rho(1,1) = \rho(2,2) = 0, \quad \rho(0,2) = 2, \rho(2,0) = 1,$$
$$\rho(1,2) = \rho(2,1) = 2, \quad \rho(0,1) = \rho(1,0) = 1.$$

Clearly, (X, ρ) is not metric space but it is a complete asymmetric metric space. Let $T : X \to X$ be given by

$$T0 = 0 = T1, \quad \text{and} \quad T2 = 1.$$

Suppose that $F_1(\alpha) = \frac{-1}{\alpha} + \alpha$, $F_2(\alpha) = \ln \alpha$ and $\tau \in (0, \frac{2}{5})$.

First observe that

$$\max\{\rho(Tx, Ty), \rho(Ty, Tx)\} > 0 \Leftrightarrow [(x = 0, y = 2) \vee (x = 1, y = 2)].$$

For $x = 0, y = 2$ we have

$$\frac{\rho(T0, T2) + \rho(T2, T0)}{2} = \frac{\rho(0,1) + \rho(1,0)}{2} = 1, \quad \frac{\rho(0,2) + \rho(2,0)}{2} = \frac{3}{2}. \tag{3.13}$$

Therefore

$$\begin{aligned} F_1\left(\frac{\rho(T0, T2) + \rho(T2, T0)}{2}\right) &= 0, \quad F_1\left(\frac{\rho(0,2) + \rho(2,0)}{2}\right) = \frac{5}{6}, \\ F_2\left(\frac{\rho(T0, T2) + \rho(T2, T0)}{2}\right) &= 0, \quad F_2\left(\frac{\rho(0,2) + \rho(2,0)}{2}\right) = \ln \frac{3}{2}. \end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\begin{aligned} \tau + F_1\left(\frac{\rho(T0, T2) + \rho(T2, T0)}{2}\right) &\leqslant F_1\left(\frac{\rho(0,2) + \rho(2,0)}{2}\right), \\ \tau + F_2\left(\frac{\rho(T0, T2) + \rho(T2, T0)}{2}\right) &\leqslant F_2\left(\frac{\rho(0,2) + \rho(2,0)}{2}\right). \end{aligned}$$

For $x = 1, y = 2$ we have

$$\frac{\rho(T1, T2) + \rho(T2, T1)}{2} = \frac{\rho(0,1) + \rho(1,0)}{2} = 1, \quad \frac{\rho(1,2) + \rho(2,1)}{2} = 2. \tag{3.15}$$

Therefore

$$\begin{aligned} F_1\left(\frac{\rho(T1, T2) + \rho(T2, T1)}{2}\right) &= 0, \quad F_1\left(\frac{\rho(1,2) + \rho(2,1)}{2}\right) = \frac{3}{2}, \\ F_2\left(\frac{\rho(T1, T2) + \rho(T2, T1)}{2}\right) &= 0, \quad F_2\left(\frac{\rho(1,2) + \rho(2,1)}{2}\right) = \ln 2. \end{aligned} \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned} \tau + F_1\left(\frac{\rho(T1, T2) + \rho(T2, T1)}{2}\right) &\leqslant F_1\left(\frac{\rho(1,2) + \rho(2,1)}{2}\right), \\ \tau + F_2\left(\frac{\rho(T1, T2) + \rho(T2, T1)}{2}\right) &\leqslant F_2\left(\frac{\rho(1,2) + \rho(2,1)}{2}\right). \end{aligned}$$

Hence T is an F_1-contraction of type (A) and F_2-contraction of type (B).

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