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Oscillation of third-order quasilinear neutral dynamic equations on time scales with distributed deviating arguments

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Abstract

The aim of this paper is to give oscillation criteria for the third-order quasilinear neutral delay dynamic equation

$$\left[r(t)\left([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+\int_c^d q_1(t)x^{\alpha}(\tau_1(t,\xi))\Delta\xi+\int_c^d q_2(t)x^{\beta}(\tau_2(t,\xi))\Delta\xi=0,$$

on a time scale $\mathbb{T}$, where $0<\alpha<\gamma<\beta$. By using a generalized Riccati transformation and integral averaging technique, we establish some new sufficient conditions which ensure that every solution of this equation oscillates or converges to zero.

Keywords: Oscillation, third order quasilinear neutral dynamic equation with distributed deviating arguments, time scales.
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1. Introduction

In this paper, we deal with the oscillatory behavior of all solutions of the third-order quasilinear neutral dynamic equation with distributed deviating arguments

$$\left[r(t)\left([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+\int_c^d q_1(t)x^{\alpha}(\tau_1(t,\xi))\Delta\xi+\int_c^d q_2(t)x^{\beta}(\tau_2(t,\xi))\Delta\xi=0, \tag{1.1}$$

on a time scale $\mathbb{T}$. In the sequel we will assume that the following conditions are satisfied:

(h1) γ,α,β are the ratio of positive odd integers such that $0<\alpha<\gamma<\beta$;

(h2) $r:\mathbb{T}\to(0,\infty)$ is a real-valued rd-continuous function on $\mathbb{T}$ and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty,\quad t_0\in\mathbb{T}; \tag{1.2}$$

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(h3) q_1, q_2 are rd-continuous positive functions on $\mathbb{T}$ and $p(t)$ is real-valued rd-continuous positive function on $\mathbb{T}$, $0 \leqslant p(t) \leqslant P < 1$;

(h4) $0 < c < d$, $\tau_0 : \mathbb{T} \to \mathbb{T}$, is rd-continuous function such that $\tau_0(t) \leqslant t$ and $\lim_{t\to\infty} \tau_0(t) = \infty$;

(h5) $\tau_i(t,\xi) : \mathbb{T} \times [c,d] \to \mathbb{T}$ are rd-continuous functions such that decreasing with respect to ξ, $\tau_i(t,\xi) \leqslant t$, $\xi \in [c,d]_{\mathbb{T}} = \{t \in \mathbb{T} : c \leqslant t \leqslant d\}$, and $\lim_{t\to\infty} \min_{\xi\in[c,d]} \tau_i(t,\xi) = \infty$ for i=1, 2 and there exists a function $\tau : \mathbb{T} \to \mathbb{T}$ which satisfies that $\tau(t) \leqslant \tau_1(t,\xi)$, $\tau(t) \leqslant \tau_2(t,\xi)$.

Define the function by

$$z(t) = x(t) + p(t)x(\tau_0(t)). \tag{1.3}$$

Furthermore, the equation (1.1) can be written as

$$\left[r(t)\left([z(t)]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta} + \int_c^d q_1(t)x^{\alpha}(\tau_1(t,\xi))\Delta\xi + \int_c^d q_2(t)x^{\beta}(\tau_2(t,\xi))\Delta\xi = 0.$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \bigcap \mathbb{T}$.

By a solution of equation (1.1), we mean a function $x \in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $T_x \in [t_0,\infty)_{\mathbb{T}}$, which has the properties $z \in C^2_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, $r(z^{\Delta\Delta})^{\gamma} \in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \in [T,\infty)_{\mathbb{T}}\} > 0$ for all $T \in [T_x,\infty)_{\mathbb{T}}$ and assume that (1.1) possesses such solutions. It is easy to see that all solutions of Eq. (1.1) can be extended to ∞ all $t \in \mathbb{T}$ or $\mathbb{T}$ is a discrete time scale. However, Eq. (1.1) may have both extendable solutions and nonextendable solutions in general. For the asymptotic and oscillation purposes, we are only interested in the solutions that are extendable to ∞.

A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [9], in order to unify continuous and discrete analysis. Since then, several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [1]. A book on the subject of time scales by Bohner and Peterson [3] also summarizes and organizes much of the time scale calculus. In the recent years, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of various equations on time-scales; we refer the reader to the papers [2–5].

To the best of our knowledge, it seems to have few oscillation results for the oscillation of third-order dynamic equations. Candan [5] studied asymptotic properties of solutions of third-order nonlinear neutral dynamic equations

$$\left(r_2(t)\left[\left(r_1(t)\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + f(t,y(\delta(t))) = 0. \tag{1.4}$$

Li et al. [11] considered third-order nonlinear delay dynamic equation

$$x^{\Delta^3} + p(t)x^{\gamma}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers.

Li et al. [10] considered third-order nonlinear delay dynamic equation

$$(a(t)([r(t)x^{\Delta}(t)]^{\Delta})^{\gamma})^{\Delta} + f(t,x(\tau(t))) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma > 0$ is quotient of odd positive integers.

Saker and Graef [12] and Zhang [15] considered a third order half-linear neutral dynamic equation

$$\left(r_1(t)((r_2(t)(x(t)+a(t)x(\tau(t)))^{\Delta})^{\Delta})^{\gamma})\right)^{\Delta}+p(t)x^{\gamma}(\delta(t))=0.$$

Han et al. [7] and Grace et al. [6] considered third-order neutral delay dynamic equation

$$(r(t)(x(t)-a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta}+p(t)x^{\gamma}(\delta(t))=0.$$

Şenel and Utku [13]- [14] considered a third order dynamic equations

$$\left[r(t)\left([x(t)+p(t)x(\tau(t))]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+\int_c^d f(t,x[\phi(t,\xi)])\Delta\xi=0,$$

$$\left[r(t)\left([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta}\right)^{\gamma}\right]^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0.$$

on a time scale $\mathbb{T}$.

In this paper, we consider third-order quasilinear neutral delay dynamic equation on time scales which is not in literature. We obtain some conclusions which contribute to oscillation theory of third order quasilinear neutral dynamic equations with distributed deviating arguments.

2. Several lemmas

Before stating our main results, we begin with the following lemmas which play an important role in the proof of the main results. Throughout this paper, we let

$$\begin{aligned}
\eta_+(t) &:= \max\{0,\eta(t)\}, \quad \eta_-(t) := \max\{0,-\eta(t)\},\\
\varphi &:= \min\{\frac{\beta-\alpha}{\beta-\gamma},\frac{\beta-\alpha}{\gamma-\alpha}\},\\
\kappa &:= \min\{k^{\alpha},k^{\beta}\},\\
\Phi(t) &= \varphi(q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{\tau(t)}{\sigma(t)})^{\gamma},\\
\beta(t) &:= \frac{t}{\sigma(t)},\ 0<\gamma\leqslant 1, \quad \beta(t) := (\frac{t}{\sigma(t)})^{\gamma},\ \gamma>1,\\
R(t,t_*) &:= \int_{t_*}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s,
\end{aligned}$$

for sufficiently large $t_*\in[t_0,\infty)_{\mathbb{T}}$.

Lemma 2.1. *Let $x(t)$ be a positive solution of (1.1), and $z(t)$ is defined as in (1.3). Then $z(t)$ has only one of the following two properties:*

(1) $z(t)>0$, $z^{\Delta}(t)>0$, $z^{\Delta\Delta}(t)>0$;
(2) $z(t)>0$, $z^{\Delta}(t)<0$, $z^{\Delta\Delta}(t)>0$,

where $t\geqslant t_1$, t_1 sufficiently large.

Proof. Let $x(t)$ be a positive solution of (1.1) on $[t_0,\infty)$, so that $z(t)>x(t)>0$, and

$$\left[r(t)(z^{\Delta\Delta}(t))^{\gamma}\right]^{\Delta}=-\int_c^d q_1(t)x^{\alpha}(\tau_1(t,\xi))\Delta\xi-\int_c^d q_2(t)x^{\beta}(\tau_2(t,\xi))\Delta\xi<0.$$

Then $r(t)([z(t)]^{\Delta\Delta})^{\gamma}$ is a decreasing function and therefore eventually of one sign, so $z^{\Delta\Delta}(t)$ is either eventually positive or eventually negative on $t \geqslant t_1 \geqslant t_0$. We assert that $z^{\Delta\Delta}(t) > 0$ on $t \geqslant t_1 \geqslant t_0$. Otherwise, assume that $z^{\Delta\Delta}(t) < 0$, then there exists a constant $M > 0$, such that

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} \leqslant -M < 0.$$

By integrating the last inequality from t_1 to t, we obtain

$$z^{\Delta}(t) \leqslant z^{\Delta}(t_1) - M^{\frac{1}{\gamma}} \int_{t_1}^{t} (\frac{1}{r(s)})^{\frac{1}{\gamma}} \Delta s.$$

Let $t \to \infty$. Then from (1.4) , we have $(z(t))^{\Delta} \to -\infty$, and therefore eventually $z^{\Delta}(t) < 0$.

Since $z^{\Delta\Delta}(t) < 0$ and $z^{\Delta}(t) < 0$, we have $z(t) < 0$, which contradicts our assumption $z(t) > 0$. Therefore, z(t) has only one of the two properties (1) and (2). This completes the proof. □

Lemma 2.2. *Let x(t) be a positive solution of* (1.1)*, correspondingly* $z(t)$ *has the property (2). If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left[\frac{1}{r(u)} \int_{u}^{\infty} (q_1(s) + q_2(s)) \Delta s \right]^{\frac{1}{\gamma}} \Delta u \Delta v = \infty, \tag{2.1}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof. Let $x(t)$ be a positive solution of (1.1). Since $z(t)$ has the property (2), then there exists finite $\lim_{t\to\infty} z(t) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$, then for any $\epsilon > 0$, we have $\ell + \epsilon > z(t) > \ell$, eventually. Choosing $0 < \epsilon < \frac{\ell(1-p)}{p}$, we obtain from (1.3)

$$x(t) = z(t) - p(t)x(\tau_0(t)) > \ell - p(t)z(\tau_0(t)) > \ell - p(t)(\ell + \epsilon) = k(\ell + \epsilon) > kz(t),$$

where $k = \frac{\ell - P(1+\epsilon)}{\ell + \epsilon} > 0$. Using (1.4), (h1), and (h5), we obtain

$$\begin{aligned} [r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} &= -\int_{c}^{d} q_1(t) x^{\alpha}(\tau_1(t,\xi))\Delta\xi - \int_{c}^{d} q_2(t) x^{\beta}(\tau_2(t,\xi))\Delta\xi \\ &< -q_1(t)k^{\alpha} \int_{c}^{d} z^{\alpha}(\tau_1(t,\xi))\Delta\xi - q_2(t)k^{\beta} \int_{c}^{d} z^{\beta}(\tau_2(t,\xi))\Delta\xi. \end{aligned}$$

Since $z^{\Delta}(t) < 0$, we have

$$[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} \leqslant -q_1(t)k^{\alpha} z^{\alpha}(\tau_1(t,d)) - q_2(t)k^{\beta} z^{\beta}(\tau_2(t,d)) \leqslant -q_1(t)\kappa z^{\alpha}(t) - q_2(t)\kappa z^{\beta}(t).$$

Then

$$[r(t)(z^{\Delta\Delta}(t))^{\gamma}]^{\Delta} \leqslant -\kappa z^{\alpha}(t)(q_1(t) + q_2(t)). \tag{2.2}$$

Integrating inequality (2.2) from t to ∞, we obtain

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} \geqslant \kappa \int_{t}^{\infty} z^{\alpha}(s)(q_1(s) + q_2(s))\Delta s.$$

Using $z^{\alpha}(s) \geqslant \ell^{\alpha}$, we obtain

$$z^{\Delta\Delta}(t) \geqslant \frac{\kappa^{1/\gamma}\ell^{\alpha/\gamma}}{r^{\frac{1}{\gamma}}(t)} \left[\int_{t}^{\infty} (q_1(s) + q_2(s))\Delta s \right]^{\frac{1}{\gamma}}. \tag{2.3}$$

Integrating inequality (2.3) from t to ∞, we have

$$-z^{\Delta}(t) \geqslant \kappa^{1/\gamma}\ell^{\alpha/\gamma}\int_t^{\infty}[\frac{1}{r(u)}\int_u^{\infty}(q_1(s)+q_2(s))\Delta s]^{\frac{1}{\gamma}}\Delta u.$$

Integrating the last inequality from t_1 to ∞, we obtain

$$z(t_1) \geqslant \kappa^{1/\gamma}\ell^{\alpha/\gamma}\int_{t_1}^{\infty}\int_v^{\infty}[\frac{1}{r(u)}\int_u^{\infty}(q_1(s)+q_2(s))\Delta s]^{\frac{1}{\gamma}}\Delta u\Delta v.$$

The last inequality contradict (2.1), we have $\ell = 0$. And since $0 \leqslant x(t) \leqslant z(t)$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3. *Assume that $x(t)$ is a positive solution of equation (1.1), and $z(t)$ is defined as in (1.3) such that $z^{\Delta\Delta}(t) > 0, z^{\Delta}(t) > 0$, on $[t_*,\infty)_{\mathbb{T}}$, $t_* \geqslant 0$. Then*

$$z^{\Delta}(t) \geqslant R(t,t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t). \tag{2.4}$$

Proof. Since $r(t)(z^{\Delta\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_*,\infty)_{\mathbb{T}}$, we get for $t \in [t_*,\infty)_{\mathbb{T}}$

$$z^{\Delta}(t) > z^{\Delta}(t) - z^{\Delta}(t_*) = \int_{t_*}^{t}\frac{(r(s)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s \geqslant (r(t)(z^{\Delta\Delta}(t))^{\gamma})^{\frac{1}{\gamma}}\int_{t_*}^{t}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s$$

and, hence

$$z^{\Delta}(t) > R(t,t_*)r^{\frac{1}{\gamma}}(t)z^{\Delta\Delta}(t) \text{ on } [t_*,\infty)_{\mathbb{T}}.$$

□

Lemma 2.4. *Assume that $x(t)$ is a positive solution of equation (1.1), correspondingly $z(t)$ has the property (1), such that $z^{\Delta}(t) > 0$, $z^{\Delta\Delta}(t) > 0$, on $[t_*,\infty)_{\mathbb{T}}$, $t_* \geqslant t_0$. Furthermore,*

$$\int_{t_0}^{\infty}(q_1(s)+q_2(s))\tau^{\alpha}(s)\Delta s = \infty. \tag{2.5}$$

Then there exists a $T \in [t_*,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$z(t) > tz^{\Delta}(t),$$

$z(t)/t$ is strictly decreasing, $t \in [T,\infty)_{\mathbb{T}}$.

Proof. Let $U(t) = z(t) - tz^{\Delta}(t)$. Hence $U^{\Delta}(t) = -\sigma(t)z^{\Delta\Delta}(t) < 0$. We claim there exists a $t_1 \in [t_*,\infty)_{\mathbb{T}}$ such that $U(t) > 0$, $z(\tau(t)) > 0$ on $[t_*,\infty)_{\mathbb{T}}$. Assume not. Then $U(t) < 0$ on $[t_*,\infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t)-z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0,$$

which implies that $z(t)/t$ is strictly increasing. Pick $t_2 \in [t_1,\infty)_{\mathbb{T}}$ so that $\tau(t) \geqslant \tau(t_*)$, for $t \geqslant t_2$. Then

$$\frac{z(\tau(t))}{\tau(t)} \geqslant \frac{z(\tau(t_*))}{\tau(t_*)} = d > 0,$$

so that $z(\tau(t)) > d\tau(t)$ for $t \geqslant t_2$. By (1.3) and (h3), we obtain

$$x(t) = z(t) - p(t)x(\tau_0(t)) > z(t) - p(t)z(\tau_0(t)) \geqslant (1-p(t))z(t) \geqslant (1-P)z(t). \tag{2.6}$$

Using (1.4) and (2.6), we have

$$[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} = -\int_c^d q_1(t)x^{\alpha}(\tau_1(t,\xi))\Delta\xi - \int_c^d q_2(t)x^{\beta}(\tau_2(t,\xi))\Delta\xi$$
$$\leqslant -q_1(t)(1-P)^{\alpha}\int_c^d z^{\alpha}(\tau_1(t,\xi))\Delta\xi - q_2(t)(1-P)^{\beta}\int_c^d z^{\beta}(\tau_2(t,\xi))\Delta\xi.$$

Using (h1) and (h5), we have

$$\begin{aligned}[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} &\leqslant -q_1(t)(1-P)^{\alpha}z^{\alpha}(\tau_1(t,c)) - q_2(t)(1-P)^{\beta}z^{\beta}(\tau_2(t,c))\\ &\leqslant -q_1(t)(1-P)^{\beta}z^{\alpha}(\tau(t)) - q_2(t)(1-P)^{\beta}z^{\alpha}(\tau(t))\\ &\leqslant -(1-P)^{\beta}z^{\alpha}(\tau(t))(q_1(t)+q_2(t)).\end{aligned}$$

Now by integrating both sides of last equation from t_2 to t, we have

$$r(t)(z^{\Delta\Delta}(t))^{\gamma} - r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} + \int_{t_2}^{t}(1-P)^{\beta}(q_1(t)+q_2(t))z^{\alpha}(\tau(t))\Delta s \leqslant 0.$$

This implies that

$$r(t_2)(z^{\Delta\Delta}(t_2))^{\gamma} \geqslant \int_{t_2}^{t}(1-P)^{\beta}(q_1(s)+q_2(s))z^{\alpha}(\tau(s))\Delta s \geqslant d^{\alpha}(1-P)^{\beta}\int_{t_2}^{t}(q_1(s)+q_2(s))\tau^{\alpha}(s)\Delta s,$$

which contradicts to (2.5). So $U(t) > 0$ on $t \in [t_1,\infty)_{\mathbb{T}}$ and consequently,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \ t \in [t_1,\infty)_{\mathbb{T}},$$

and we have that $z(t)/t$ is strictly decreasing on $t \in [t_1,\infty)_{\mathbb{T}}$. The proof is now complete. □

3. Main results

In this section we give some new oscillation criteria for (1.1).

Theorem 3.1. *Assume that (1.2), (2.1), and (2.5) hold and that, for all sufficiently large $T_1 \in [t_0,\infty)_{\mathbb{T}}$, there is a $T > T_1$ such that*

$$\limsup_{t\to\infty}\int_T^t \left[\rho^{\sigma}(s)\Phi(s) - \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,t_*))^{\gamma}}\right]\Delta s = \infty, \tag{3.1}$$

where the function $\rho \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is a nonnegative function. Then every solution of equation (1.1) is either oscillatory or tends to zero.

Proof. Assume (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality we assume that $x(t) > 0$, $x(\tau_0(t)) > 0$ for $t \geqslant t_1$ and $x(\tau_1(t,\xi)) > 0$, $x(\tau_2(t,\xi)) > 0$ for $(t,\xi) \in [t_1,\infty) \times [c,d]$ for all $t_1 \in [t_0,\infty)_{\mathbb{T}}$. $z(t)$ is defined as in (1.3). We shall consider only $z(t) > 0$, since the proof when $z(t)$ is eventually negative is similar. Therefore by Lemmas 2.1 and 2.2, we have

$$\left[r(t)([z(t)]^{\Delta\Delta})^{\gamma}\right]^{\Delta} < 0, \quad z^{\Delta\Delta}(t) > 0, \quad t \in [t_1,\infty)_{\mathbb{T}},$$

and either $z^{\Delta}(t) > 0$ for $t \geqslant t_2 \geqslant t_1$ or $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} x(t) = 0$. Let $z^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Define the function $w(t)$ by Riccati substitution

$$w(t) = \rho(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}.$$

Then

$$w^{\Delta}(t) = \rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)} + \rho^{\sigma}(t)\left[\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)}\right]^{\Delta}$$
$$= \rho^{\Delta}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}}{z^{\gamma}(t)} + \rho^{\sigma}(t)\frac{[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta}}{z^{\gamma\sigma}(t)} - \rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}.$$

By (1.4) and (2.6), we have

$$[r(t)([z(t)]^{\Delta\Delta})^{\gamma}]^{\Delta} \leqslant -q_1(t)(1-P)^{\alpha}\int_c^d z^{\alpha}(\tau_1(t,\xi))\Delta\xi - q_2(t)(1-P)^{\beta}\int_c^d z^{\beta}(\tau_2(t,\xi))\Delta\xi$$
$$\leqslant -q_1(t)(1-P)^{\alpha}z^{\alpha}(\tau_1(t,c)) - q_2(t)(1-P)^{\beta}z^{\beta}(\tau_2(t,c))$$
$$\leqslant -q_1(t)(1-P)^{\beta}z^{\alpha}(\tau(t)) - q_2(t)(1-P)^{\beta}z^{\beta}(\tau(t)).$$

From the definition of $w(t)$ and the last inequality, we have,

$$\begin{aligned} w^{\Delta}(t) \leqslant{}& \frac{\rho^{\Delta}(t)}{\rho(t)}w(t) - \rho^{\sigma}(t)q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau(t))}{z^{\gamma}(\sigma(t))} - \rho^{\sigma}(t)q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau(t))}{z^{\gamma}(\sigma(t))} \\ &- \rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}. \end{aligned} \tag{3.2}$$

By Young's inequality

$$|ab| \leqslant \frac{1}{p}|a|^p + \frac{1}{q}|b|^q,\ a,b\in\mathbb{R},\ p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\begin{aligned} &\frac{\beta-\gamma}{\beta-\alpha}q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau(t))}{z^{\gamma}(\sigma(t))} + \frac{\gamma-\alpha}{\beta-\alpha}q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau(t))}{z^{\gamma}(\sigma(t))} \\ &\geqslant \left[q_1(t)(1-P)^{\alpha}\frac{z^{\alpha}(\tau(t))}{z^{\gamma}(\sigma(t))}\right]^{(\beta-\gamma)/(\beta-\alpha)}\left[q_2(t)(1-P)^{\beta}\frac{z^{\beta}(\tau(t))}{z^{\gamma}(\sigma(t))}\right]^{(\gamma-\alpha)/(\beta-\alpha)} \\ &= (q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{z^{\alpha}(\tau(t))}{z^{\gamma}(\sigma(t))})^{(\beta-\gamma)/(\beta-\alpha)}(\frac{z^{\beta}(\tau(t))}{z^{\gamma}(\sigma(t))})^{(\gamma-\alpha)/(\beta-\alpha)} \\ &\geqslant (q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{z(\tau(t)}{z(\sigma(t))})^{\gamma}. \end{aligned} \tag{3.3}$$

Hence, by (3.2) and (3.3) and using the fact that $z(t)/t$ is decreasing, we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant{}& \frac{\rho^{\Delta}(t)}{\rho(t)}w(t) - \varphi\rho^{\sigma}(t)(q_1(t)(1-P)^{\alpha})^{(\beta-\gamma)/(\beta-\alpha)}(q_2(t)(1-P)^{\beta})^{(\gamma-\alpha)/(\beta-\alpha)}(\frac{(\tau(t)}{(\sigma(t))})^{\gamma} \\ &- \rho^{\sigma}(t)\frac{r(t)([z(t)]^{\Delta\Delta})^{\gamma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma\sigma}(t)}. \end{aligned}$$

In the first case $0<\gamma\leqslant 1$. Using the Keller's chain rule (see [3]), we have

$$(z^{\gamma}(t))^{\Delta} = \gamma\int_0^1 [hz^{\sigma}+(1-h)z]^{\gamma-1}z^{\Delta}(t)dh \geqslant \gamma(z^{\sigma}(t))^{\gamma-1}z^{\Delta}(t). \tag{3.4}$$

In view of (3.4), Lemmas 2.2 and 2.3, (2.4), and using the fact that $z(t)/t$ is decreasing, we have

$$\begin{aligned} w^{\Delta}(t) &\leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)\frac{r(t)(z^{\Delta\Delta}(t))^{\gamma}z^{\Delta}(t)z(t)}{z^{\gamma+1}(t)z^{\sigma}(t)} \\ &\leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{r^{\frac{\gamma+1}{\gamma}}(t)(z^{\Delta\Delta}(t))^{\gamma+1}z(t)}{z^{\gamma+1}(t)z(\sigma(t))} \\ &\leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)}w(t) - \gamma\rho^{\sigma}(t)R(t,t_*)\frac{t}{\sigma(t)}\frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \end{aligned} \tag{3.5}$$

Let $\gamma > 1$. Applying the Keller's chain rule, we have

$$(z^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hz^{\sigma} + (1-h)z]^{\gamma-1} z^{\Delta}(t)dh \geqslant \gamma (z(t))^{\gamma-1} z^{\Delta}(t), \tag{3.6}$$

in the view of (3.6), Lemmas 2.2 and 2.3, and (2.4), we have

$$\begin{aligned} w^{\Delta}(t) &\leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)} w(t) - \gamma\rho^{\sigma}(t) \frac{r(t)[z(t)]^{\Delta\Delta})^{\gamma} z^{\Delta}(t) z^{\gamma}(t)}{z^{\gamma+1}(t) z^{\gamma\sigma}(t)}, \\ w^{\Delta}(t) &\leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)} w(t) - \gamma\rho^{\sigma}(t) (\frac{t}{\sigma(t)})^{\gamma} R(t, t_*) \frac{w^{\frac{\gamma+1}{\gamma}}(t)}{\rho^{\frac{\gamma+1}{\gamma}}(t)}. \end{aligned} \tag{3.7}$$

By (3.5), (3.7), and the definition of $\beta(t)$, we have, for $\gamma > 0$,

$$w^{\Delta}(t) \leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{(\rho^{\Delta}(t))_+}{\rho(t)} w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t, t_*) \frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}, \tag{3.8}$$

where $\lambda := \frac{\gamma+1}{\gamma}$.

Define $A \geqslant 0$ and $B \geqslant 0$ by

$$\begin{aligned} A^{\lambda} &:= \gamma\rho^{\sigma}(t)\beta(t)R(t, t_*) \frac{w^{\lambda}(t)}{\rho^{\lambda}(t)}, \\ B^{\lambda-1} &:= \frac{\rho^{\Delta}(t)}{\lambda(\gamma\rho^{\sigma}(t)\beta(t)R(t, t_*))^{\frac{1}{\lambda}}}. \end{aligned}$$

Then using the inequality [8]

$$\lambda A B^{\lambda-1} - A^{\lambda} \leqslant (\lambda-1) B^{\lambda},$$

which yields

$$\frac{(\rho^{\Delta}(t))_+}{\rho(t)} w(t) - \gamma\rho^{\sigma}(t)\beta(t)R(t, t_*) \frac{w^{\lambda}(t)}{\rho^{\lambda}(t)} \leqslant \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t, t_*))^{\gamma}}.$$

From this last inequality and (3.8), we find

$$w^{\Delta}(t) \leqslant -\rho^{\sigma}(t)\Phi(t) + \frac{((\rho^{\Delta}(t))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(t)\rho^{\sigma}(t)R(t, t_*))^{\gamma}}.$$

Integrating both sides from T to t, we get

$$\int_T^t [\rho^{\sigma}(s)\Phi(s) - \frac{((\rho^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s, t_*))^{\gamma}}]\Delta s \leqslant w(T) - w(t) \leqslant w(T),$$

for all large t, which contradicts to assumption (3.1). If (2) holds, from Lemma 2.2, then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Theorem 3.2. *Assume that* (1.2), (2.1), *and* (2.5) *hold. Furthermore, suppose that there exist functions* $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv (t, s) : t \geqslant s \geqslant t_0$ *such that*

$$\begin{aligned} &H(t, t) = 0, \ t \geqslant 0, \\ &H(t, s) > 0, \ t > s \geqslant t_0, \end{aligned}$$

and H has a nonpositive continuous Δ-partial derivative $H^{\Delta s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta s}(\sigma(t),s)+H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}=-\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}, \tag{3.9}$$

and for all sufficiently large $T_1\in[t_0,\infty)_{\mathbb{T}}$, there is a $T>T_1$ such that

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)}\chi(t,s)\Delta s=\infty, \tag{3.10}$$

where ρ is a positive Δ-differentiable function and

$$\chi(t,s)=H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)-\frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^{\sigma}(s)R(s,T_1))^{\gamma}}.$$

Then every solution of equation (1.1) is either oscillatory or tends to zero.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) and $z(t)$ is defined as in (1.2). Without loss of generality, we may assume that there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 2.1 hold and (3.9) holds for $t_2>t_1$. If case (1) of Lemma 2.1 holds, then proceeding as in the proof of Theorem 3.1, we see that (3.8) holds for $t>t_2$. Multiplying both sides of (3.8) by $H(\sigma(t),\sigma(s))$ and integrating from T to $\sigma(t)$, we get

$$\begin{aligned}\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)\Delta s\leqslant&-\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s+\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}w(s)\Delta s\\&-\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s,\ \left(\lambda=\frac{\gamma+1}{\gamma}\right).\end{aligned} \tag{3.11}$$

Integrating by parts and using $H(t,t)=0$, we obtain

$$\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s=-H(\sigma(t),T)w(T)-\int_T^{\sigma(t)}H^{\Delta s}(\sigma(t),s)w(s)\Delta s.$$

It then follows from (3.11) that

$$\begin{aligned}\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)\Delta s\leqslant&\ H(\sigma(t),T)w(T)+\int_T^{\sigma(t)}H^{\Delta s}(\sigma(t),s)w(s)\Delta s\\&+\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}w(s)\Delta s\\&-\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)\Delta s\leqslant&\ H(\sigma(t),T)w(T)+\left[\int_T^{\sigma(t)}H^{\Delta s}(\sigma(t),s)+H(\sigma(t),\sigma(s))\frac{\rho^{\Delta}(s)}{\rho(s)}\right]w(s)\Delta s\\&-\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.\end{aligned}$$

It then follows from (3.9) that

$$\begin{aligned}\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\rho^{\sigma}(s)\Phi(s)\Delta s\leqslant&\ H(\sigma(t),T)w(T)+\int_T^{\sigma(t)}\left[-\frac{h(t,s)}{\rho(s)}H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\right]w(s)\Delta s\\&-\int_T^{\sigma(t)}H(\sigma(t),\sigma(s))\gamma\rho^{\sigma}(s)\beta(s)R(s,T_1)\frac{w^{\lambda}(s)}{\rho^{\lambda}(s)}\Delta s.\end{aligned}$$

Then

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \leqslant H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} \left[\frac{h_-(t,s)}{\rho(s)} H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\right] w(s)\Delta s \\ - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s. \tag{3.12}$$

Therefore, as in Theorem 3.1, by letting

$$A^\lambda := H(\sigma(t),\sigma(s))\gamma\rho^\sigma(t)\beta(t)R(t,T_1)\frac{w^\lambda(t)}{\rho^\lambda(t)},$$
$$B^{\lambda-1} := \frac{h_-(t,s)}{\lambda(\gamma\rho^\sigma(t)\beta(t)R(t,T_1))^{\frac{1}{\lambda}}},$$

then using the inequality [8]

$$\lambda A B^{\lambda-1} - A^\lambda \leqslant (\lambda-1)B^\lambda,$$

we have

$$\int_T^{\sigma(t)} \left[\frac{h_-(t,s)}{\rho(s)} H(\sigma(t),\sigma(s))^{\frac{\gamma}{\gamma+1}}\right] w(s)\Delta s - \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\gamma\rho^\sigma(s)\beta(s)R(s,T_1)\frac{w^\lambda(s)}{\rho^\lambda(s)}\Delta s \\ \leqslant \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s.$$

From this last inequality and (3.12), we find

$$\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s \leqslant H(\sigma(t),T)w(T) + \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(t,T_1))^\gamma}\Delta s.$$

Then for $T > T_1$ we have

$$\int_T^{\sigma(t)} \left[H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s) - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\right]\Delta s \leqslant H(\sigma(t),T)w(T),$$

and this implies that

$$\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} \left[H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s) - \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\right]\Delta s < w(T),$$

for all large t, which contradicts (3.10). This completes the proof of Theorem 3.2. □

Remark 3.3. From Theorem 3.1, we can obtain different conditions for oscillation of equation (1.1) with different choices of $\rho(t)$.

Remark 3.4. The conclusion of Theorem 3.1 remains intact if assumption (3.1) is replaced by the two conditions

$$\limsup_{t\to\infty} \int_T^t \rho^\sigma(s)\Phi(s)\Delta s = \infty,$$
$$\limsup_{t\to\infty} \int_T^t \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\Delta s < \infty.$$

Remark 3.5. The conclusion of Theorem 3.2 remains intact if assumption (3.10) is replaced by the two conditions

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\rho^\sigma(s)\Phi(s)\Delta s = \infty,$$
$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^{\sigma(t)} \frac{(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,T_1))^\gamma}\Delta s < \infty.$$

Example 3.6. Consider the third order quasilinear neutral dynamic equations on time scales with distributed deviating arguments

$$\left(x(t) + \frac{1}{3}x(\tau_0(t))\right)^{\Delta\Delta\Delta} + \int_c^d \frac{1}{t}x^{\frac{1}{3}}(\frac{t}{2})\Delta t + \int_c^d \frac{1}{t}x^{\frac{5}{3}}(\frac{t}{2})\Delta t = 0, \tag{3.13}$$

where $r(t) = 1,\ \alpha = \frac{1}{3},\ \gamma = 1,\ \beta = \frac{5}{3},\ q_1(t) = q_2(t) = \frac{1}{t}$, and μ is a positive constant.

The conditions (1.2), (2.1), and (2.5) hold. By Theorem 3.1, pick $\rho(t) = t$, we have

$$\limsup_{t\to\infty} \int_T^t \left[\rho^\sigma(s)\Phi(s) - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s = \limsup_{t\to\infty} \int_T^t \left[\mu - \frac{1}{s(s-t_*)}\right]\Delta s = \infty.$$

Hence, every solution of Eq. (3.13) is oscillatory or tends to zero if $\mu > 0$.

Example 3.7. Consider the third order quasilinear neutral dynamic equations on time scales with distributed deviating arguments

$$\left(\frac{1}{t^2}(x(t) + \frac{1}{3}x(\tau_0(t)))\right)^{\Delta\Delta\Delta} + \int_c^d \frac{\sigma(t)}{\tau(t)}x^{\frac{1}{3}}(\tau_1(t))\Delta t + \int_c^d \frac{\sigma(t)}{\tau(t)}x^{\frac{5}{3}}(\tau_2(t))\Delta t = 0, \tag{3.14}$$

where $r(t) = \frac{1}{t^2},\ \alpha = \frac{1}{3},\ \gamma = 1,\ \beta = \frac{5}{3},\ q_1(t) = q_2(t) = \frac{\sigma(t)}{\tau(t)}$, and μ is a positive constant.

The conditions (1.2), (2.1), and (2.5) hold. By Theorem 3.1, pick $\rho(t) = 1$, we have

$$\limsup_{t\to\infty} \int_T^t \left[\rho^\sigma(s)\Phi(s) - \frac{((\rho^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\beta(s)\rho^\sigma(s)R(s,t_*))^\gamma}\right]\Delta s = \limsup_{t\to\infty} \int_T^t \mu\Delta s = \infty.$$

Hence, every solution of Eq. (3.14) is oscillatory or tends to zero if $\mu > 0$.

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