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# THREE-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD FOR SOLVING NONLINEAR THREE-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

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## 1. Introduction

During the last 15 years, numerical analysis of one-dimensional Volterra Integral equations has been discussed in [2] (and in the references cited there). The numerical methods for two-dimensional Volterra Integral equation seem to have been studied in some places. Ref. [1] proposed a class of explicit Runge-Kutta type methods of order 3 (without analyzing their convergence). Brunner and Kauthen [3] introduced collocation and iterated collocation methods for two-dimensional linear Volterra integral equation. Guo-qiang and Hayami [4] introduced extrapolation method of iterated collocation solution for two-dimensional nonlinear Volterra integral equation. Recently in [5] P. Darnia and A. Ebadian introduced the numerical solution of non-linear Volterra integral equations.

The subject of the presented paper is to apply three-dimensional (DTM) for solving nonlinear three-dimensional Volterra equations. For this purpose we will consider-nonlinear three-dimensional Volterra equation of the second kind.

**1.1** $$u(x,y,z) = g(x,y,z) + \int_0^x \int_0^y \int_0^z \emptyset(x,y,z,r,s,t,u)\,dt\,ds\,dr$$

where

$$(x,y,z) \in D = [0,X] \times [0,Y] \times [0,Z], u(x,y,z)$$

is an unknown function, $g(x,y,z)$ and $\emptyset(x,y,z,r,s,t,u)$ are given analytical functions defined, respectively on D and

$$F = \{(x,y,z,r,s,t,u) : 0 \le r \le x \le X\ , 0 \le r \le y \le Y\ , 0 \le t \le z \le Z\ , 0 < u < \infty\}.$$

Order to solve three-dimensional non-linear Volterra equation by differential transform, its basic theory is stated in next section.

# 2. Main Result

**Definition 2.1.** Consider the analytical function of three variables $u(x,y,z)$ which is defined on $D \subset \mathcal{R}^3$ and $(x_0,y_0,z_0) \in D$. Three dimensional Differential Transform method of $u(x,y,z)$ is denoted by $U(k,h,l)$ is defined on $\mathbb{N}^3 \cup (0,0,0)$, as following

2.1 $$U(k,h,l) = \frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k \partial y^h \partial z^l}\right]_{(x_0,y_0,z_0)},$$

where $u(x,y,z)$ is original function and $U(k,h,l)$ is called transformed function. Inverse differential transform $U(k,h,l)$ in Eq(2.1) is defined as following

2.2 $$u(x,y,z) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{l=0}^{\infty} U(k,h,l)(x-x_0)^k(y-y_0)^h(z-z_0)^l.$$

By combining Eqs(2.1) and (2.2) with $(x_0, y_0, z_0) = (0,0,0)$, the function $u(x, y, z)$ can be written as

2.3 $$u(x,y,z) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k\partial y^h\partial z^l}\right]x^k y^h z^l.$$

In real applications, the function $u(x, y, z)$ is expressed by finite series and Eq(2.3) can bewritten as following

2.4 $$u(x,y,z) = \sum_{k=0}^{p}\sum_{h=0}^{q}\sum_{l=0}^{n}\frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k\partial y^h\partial z^l}\right]x^k y^h z^l.$$

This equation implies that

2.5 $$u(x,y,z) = \sum_{k=p+1}^{\infty}\sum_{h=q+1}^{\infty}\sum_{l=n+1}^{\infty}\frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k\partial y^h\partial z^l}\right]x^k y^h z^l,$$

is negligibly small. The fundamental mathematical properties of three-dimensional differential transform are pressed in the following theorem.

**Theorem 2.2.** If $U(k,h,l), F(k,h,l)$ and $G(k,h,l)$ are three-dimensional differential transform of the functions $u(x,y,z), f(x,y,z), g(x,y,z)$ respectively, then

1. If $u(x,y,z) = f(x,y,z) + g(x,y,z)$, then
$$U(k,h,l) = F(x,y,z) + G(x,y,z).$$
2. If $u(x,y,z) = x^p y^q z^r$, then $U(k,h,l) = \delta(k-p)\delta(h-q)\delta(l-r)$.
3. If $u(x,y,z) = \frac{\partial^{p+q+r}f(x,y,z)}{\partial x^p\partial y^q\partial z^r}$, then
$$U(k,h,l) = (k+1)\dots(k+p)(h+1)\dots(h+q)(l+1)\dots(l+r)F(k+p,+q,l+r).$$
4. If $u(x,y,z) = \sin(ax+by+cz)$, then
$$U(k,h,l) = \frac{a^k b^h c^l}{k!h!l!}\sin(\frac{k+h+l}{2}\pi).$$
5. If $u(x,y,z) = e^{ax+by+cz}$then $U(k,h,l) = \frac{a^k b^h c^l}{k!h!l!}$.

**Theorem 2.3.** If $u(x,y,z) = f(x,y,z)g(x,y,z)$ and $U(k,h,l), F(k,h,l)$ and $G(k,h,l)$ are as Theorem 2.2, then

$$U(k,h,l) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}F(r,s,t)G(k-r,h-s,l-t).$$

*Proof.* It is easy to verify that for every $k, h$ and $l \in \mathbb{N}\cup\{0\}$, we have

2.6 $$\frac{\partial^{k+h+l}(fg)(x,y,z)}{\partial x^k \partial y^h \partial z^l} =$$

$$\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}\binom{k}{r}\binom{h}{s}\binom{l}{t}\frac{\partial^{r+s+t}f(x,y,z)}{\partial x^r \partial y^s \partial z^t}\frac{\partial^{k-r+h-s+l-t}g(x,y,z)}{\partial x^{k-r}\partial y^{h-s}\partial z^{l-t}}$$

Now by using relation (2.4) and Definition 2.1, with $(x_0, y_0, z_0) = (0,0,0)$, we have

$$U(k,h,l) = \frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k \partial y^h \partial z^l}\right] =$$

$$\frac{1}{k!h!l!}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}\binom{k}{r}\binom{h}{s}\binom{l}{t}\frac{\partial^{r+s+t}f(x,y,z)}{\partial x^r \partial y^s \partial z^t}\frac{\partial^{k-r+h-s+l-t}g(x,y,z)}{\partial x^{k-r}\partial y^{h-s}\partial z^{l-t}} =$$

$$\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}\frac{1}{r!s!t!}\frac{\partial^{r+s+t}f(x,y,z)}{\partial x^r \partial y^s \partial z^t}\frac{1}{(k-r)!(h-s)!(l-t)!}\frac{\partial^{k-r+h-s+l-t}g(x,y,z)}{\partial x^{k-r}\partial y^{h-s}\partial z^{l-t}},$$

and finally, we obtain

$$U(k,h,l) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}F(r,s,t)G(k-r,h-s,l-t).$$

**Theorem 2.4.** If $u(x,y,z) = f_1(x,y,z)f_2(x,y,z)\ldots f_n(x,y,z)$, then

$$U(k,h,l) = \sum_{r_{n-1}=0}^{k}\sum_{r_{n-2}=0}^{r_{n-1}}\cdots\sum_{r_1=0}^{r_2}\sum_{s_{n-1}=0}^{h}\sum_{s_{n-2}=0}^{s_{n-1}}\cdots\sum_{s_1=0}^{s_2}\sum_{t_{n-1}=0}^{l}\sum_{t_{n-2}=0}^{t_{n-1}}\cdots\sum_{t_1=0}^{t_2}[F_1(r_1,s_1,t_1)\times F_2(r_2-r_1,s_2-s_1,t_2-t_1)\times\ldots\times F_{n-1}(r_{n-1}-r_{n-2},s_{n-1}-s_{n-2},t_{n-1}-t_{n-2})F_n(k-r_{n-1},h-s_{n-1},l-t_{n-1}).$$

*Proof.* We will proof the Theorem by induction on $n$. According to the Theorem 2.3, for $n = 2$, the theorem is clear. Suppose the theorem is correct for $n \geq 3$. We set

$$g(x,y,z) = f_1(x,y,z)f_2(x,y,z)\ldots f_{n-1}(x,y,z).$$

By using Theorem 2.3 and hypothesis, we have

2.7 $$U(k,h,l) = \sum_{r_{n-1}=0}^{k}\sum_{s_{n-1}=0}^{h}\sum_{t_{n-1}=0}^{l}G(r_{n-1},s_{n-1},t_{n-1})F_n(k-r_{n-1},h-s_{n-1},l-t_{n-1}),$$

and

2.8 $$G(k,h,l) = \sum_{r_{n-2}=0}^{k}\cdots\sum_{r_1=0}^{r_2}\sum_{s_{n-2}=0}^{h}\cdots\sum_{s_1=0}^{s_2}\sum_{t_{n-2}=0}^{l}\cdots\sum_{t_1=0}^{t_2}[F_1(r_1,s_1,t_1)\times F_3(r_3-r_2,s_3-s_2,t_3-t_2)\times\ldots\times F_{n-1}(k-r_{n-2},h-s_{n-2},l-t_{n-2})].$$

The assertion is obtained by substituting Eq (2.8) in Eq (2.7).

**Theorem 2.5.** If $(x,y,z) = \int_0^x\int_0^y\int_0^z f(r,s,t)\,dtdsdr$, then

$$U(k,h,l) = \begin{cases} 0 & if\ k=0\ or\ h=0\ or\ l=0 \\ \frac{1}{khl}F(k-1,h-1,l-1) & if\ k,h,l=1,2,3 \end{cases}$$

*Proof*. By using Leibniz formula and mathematical induction on $k, h$ and $l$, we have

2.9 $$\frac{\partial^l u(x,y,z)}{\partial z^l} = \int_{x_0}^{x}\int_{y_0}^{y}\frac{\partial^{l-1}f(r,s,z)}{\partial z^{l-1}}dsdr,$$

2.10 $$\frac{\partial^h u(x,y,z)}{\partial y^h} = \int_{x_0}^{x}\int_{z_0}^{z}\frac{\partial^{h-1}f(r,y,t)}{\partial y^{h-1}}dtdr,$$

2.11 $$\frac{\partial^k u(x,y,z)}{\partial x^k} = \int_{y_0}^{y}\int_{z_0}^{z}\frac{\partial^{k-1}f(x,s,t)}{\partial x^{k-1}}dsdt.$$

Hence, by applying Eqs (2.9), (2.10) and (2.11) in Definition 2.1 with $(x_0, y_0, z_0) = (0,0,0)$, we have $U(k,h,l) = 0$ if $k = 0$ or $h = 0$ or $l = 0$, and for $k \geq 1, h \geq 1$ and $l \geq 1$, we get

$$U(k,h,l) = \frac{1}{k!h!l!}\left[\frac{\partial^{k+h+l}u(x,y,z)}{\partial x^k \partial y^h \partial z^l}\right] = \frac{1}{k!h!l!}\frac{\partial^{k+h}}{\partial x^k \partial y^h}\left[\int_{x_0}^{x}\int_{y_0}^{y}\frac{\partial^{l-1}f(r,s,z)}{\partial z^{l-1}}dsdr\right] =$$
$$\frac{1}{k!h!l!}\frac{\partial^k}{\partial x^k}\left[\int_{x_0}^{x}\frac{\partial^{h+l-2}f(r,y,z)}{\partial y^{h-1}\partial z^{l-1}}dr\right] = \frac{1}{k!h!l!}\left[\frac{\partial^{k-1+h-1+l-1}f(x,y,z)}{\partial x^{k-1}\partial y^{h-1}\partial z^{l-1}}\right].$$

Therefore by Definition 2.1 with $(x_0, y_0, z_0) = (0,0,0)$, we will have

$$U(k,h,l) = \frac{1}{khl}F(k-1,h-1,l-1).$$

**Theorem 2.6**. If $(x,y,z) = \int_0^x\int_0^y\int_0^z f_1(r,s,t)f_2(r,s,t)\ldots.f_n(r,s,t)dtdsdr$, then

$$U(k,h,l)$$
$$= \frac{1}{khl}\sum_{r_{n-1}=0}^{k-1}\sum_{r_{n-2}=0}^{r_{n-1}}\ldots\sum_{r_1=0}^{r_2}\sum_{s_{n-1}=0}^{h-1}\sum_{s_{n-2}=0}^{s_{n-1}}\ldots\sum_{s_1=0}^{s_2}\sum_{t_{n-1}=0}^{l-1}\sum_{t_{n-2}=0}^{t_{n-1}}\ldots\sum_{t_1=0}^{t_2}[F_1(r_1,s_1,t_1)$$
$$\times F_2(r_2-r_1, s_2-s_1, t_2-t_1) \times \ldots \times F_n(k-1-r_{n-1}, h-1-s_{n-1}, l-1-t_{n-1})]$$

where $k = 1,2,\ldots,p$ , $h = 1,2,\ldots,q$ and $l = 1,2,\ldots,n$ and $U(k,h,l) = 0$ if $k = 0$ or $h = 0$ or $l = 0$.

*Proof*. We set $g(r,s,t) = f_1(r,s,t)f_2(r,s,t)\ldots f_n(r,s,t)$, then

$$u(x,y,z) = \int_0^x\int_0^y\int_0^z g(r,s,t)dtdsdr.$$

The assertion is obtained by applying Theorem 2.5 and Theorem 2.4.

**Theorem 2.7.** If $u(x,y,z) = f_1(x,y,z)\int_0^x\int_0^y\int_0^z f_2(r,s,t)dtdsdr$, then

$$U(k,h,l) = \sum_{r=1}^{k}\sum_{s=1}^{h}\sum_{t=1}^{l}\left[\frac{1}{(k-r)(h-s)(l-t)}F_1(r,s,t)F_2(k-r-1,h-s-1,l-t-1)\right]$$

where $k = 1,2,\dots,p$ , $h = 1,2,\dots,q$ and $l = 1,2,\dots,n$ and $U(k,h,l) = 0$ if $k = 0$ or $h = 0$or $l = 0$.

*Proof.* We set $g(x,y,z) = \int_0^x\int_0^y\int_0^z f_2(r,s,t)dtdsdr$, then we have

$$u(x,y,z) = f_1(x,y,z)g(x,y,z).$$

By using Theorem 2.3 and Theorem 2.5, we get

2.12 $$U(k,h,l) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{t=0}^{l}F_1(r,s,t)G(k-r,h-s,l-t),$$

and we also get

2.13 $$G(k,h,l) = \frac{1}{khl}F_2(k-1,h-1,l-1),$$

where $k = 1,2,\dots,p$ , $h = 1,2,\dots,q$ and $l = 1,2,\dots,n$ and $G(k,h,l) = 0$ if $k = 0$ or $h = 0$ or $l = 0$. Now the assertion is obtained by substituting relation (2.13), in relation 2.12.

# 3. Error Analysis

In this section, we perform the estimating error for the integral equations. Since the truncated Taylor series or the corresponding polynomial expansion is an approximate solution of equation(1.1), if we define $e_{p,q,n}(x,y,z)$as an error function in the following form

$$e_{p,q,n}(x,y,z) = \left|u(x,y,z) - u_{p,q,n}(x,y,z)\right|.$$

Where

$$u_{p,q,n}(x,y,z) = \sum_{k=0}^{p}\sum_{h=0}^{q}\sum_{l=0}^{n}U(k,h,l)(x-x_0)^k(y-y_0)^h(z-z_0)^l,$$

is approximation function, then we can prescribe $e_{p,q,n}(x,y,z) \le 10^m$, where $m$ is any positiveinteger, then we increase p and q as far as the following inequality holds at each points $(x,y,z)$

$$e_{p,q,n}(x,y,z) \le 10^m.$$

In other words, by increasing $p, q$ and $n$, the error function $e_{p,q,n}(x, y, z)$ approaches to zero.

**Example 3.1.** Consider the nonlinear Volterra Integral equation

$$u(x, y, z) = g(x, y, z) - \int_0^x \int_0^y \int_0^z u(r, s, t) dt ds dr,$$

where $(x, y, z) \in [0,1] \times [0,1] \times [0,1]$ and

$$g(x, y, z) = \frac{x^2yz + xy^2z + xyz^2}{2} + x + y + z.$$

Its exact solution can be expressed as $u(x, y, z) = x + y + z$.

We set $f(x, y, z) = \int_0^x \int_0^y \int_0^z u(r, s, t) dt ds dr$, then we have

$$u(x, y, z) = g(x, y, z) - f(x, y, z).$$

Taking the transformation of this equation , we obtain

$$U(k, h, l) = G(k, h, l) - F(k, h, l),$$

where

$$G(k, h, l) = \frac{1}{2}(\delta(k-2)\delta(h-1)\delta(l-1) + \delta(k-1)\delta(h-2)\delta(l-1) + \delta(k-1)\delta(h-1)\delta(l-2)) + \delta(k-1)\delta(h)\delta(l) + \delta(k)\delta(h-1)\delta(l) + \delta(k)\delta(h)\delta(l-1),$$

for $k = 0,1, \dots, p$, $h = 0, 1, \dots, q$ and $l = 0,1, \dots, n$, and

$$F(k, h, l) = \frac{1}{khl} U(k-1, h-1, l-1), k \geq 1, h \geq 1, l \geq 1.$$

For $p = 1, q = 2$, $l = 3$, and recursive method, we obtain

$$\begin{bmatrix} U(0,0,0) & U(0,01) & U(0,0,2) & U(0,0,3) \\ U(0,1,0) & U(0,1,1) & U(0,1,2) & U(0,1,3) \\ U(0,2,0) & U(0,2,1) & U(0,2,2) & U(0,2,3) \\ U(1,0,0) & U(1,01) & U(1,0,2) & U(1,0,3) \\ U(1,1,0) & U(1,1,1) & U(1,1,2) & U(1,1,3) \\ U(1,2,0) & U(1,2,1) & U(1,2,2) & U(1,2,3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now by substituting this relation in equation (2.4), we get

$$u(x, y, z) = x + y + z.$$

**Example 3.2.** Consider nonlinear Volterra Integral equation

$$u(x, y, z) = g(x, y, z) - 24x^2y \int_0^x \int_0^y \int_0^z u(r, s, t) dt ds dr,$$

where $(x, y, z) \in [0,1] \times [0,1] \times [0,1]$ and

$$g(x, y, z) = 4x^5y^3z + 4x^3y^3z^3 + 4x^4y^3z^2 + x^2y + yz^2 + xyz.$$

Its exact solution can be expressed as $u(x, y, z) = x^2y + yz^2 + xyz$.

We set $f(x,y,z) = -24x^2y\int_0^x\int_0^y\int_0^z u(r,s,t)dtdsdr$, then we have

$$u(x,y,z) = g(x,y,z) + f(x,y,z).$$

If we take the transformation ofthis equation, we obtain

$$U(k,h,l) = G(k,h,l) + F(k,h,l),$$

where

$$G(k,h,l) = 4\delta(k-5)\delta(h-3)\delta(l-1) + 4\delta(k-3)\delta(h-3)\delta(l-3) + 4\delta(k-4)\delta(h-3)\delta(l-2) + \delta(k-2)\delta(h-1)\delta(l) + \delta(k)\delta(h-1)\delta(l-2) + \delta(k-1)\delta(h-1)\delta(l-1),$$

for $k = 0,1,\ldots,p, h = 0,1,\ldots,q$ and $l = 0,1,\ldots,n$, and

$$F(k,h,l) = -24\sum_{r=0}^{p}\sum_{s=0}^{q}\sum_{t=0}^{n}\frac{1}{(k-r)(h-s)(l-t)}\delta(k-2)\delta(h-1)\delta(l) \times U(k-r-1,h-s-1,l-t-1),$$

for $k = 1,\ldots,p, h = 1,\ldots,q$ and $l = 1,\ldots,n$.

By solving the above recursive equations for $p = 3, q = 1$ and $n = 3$, the result are listed as follows

$$\begin{bmatrix} U(0,0,0) & U(0,01) & U(0,0,2) & U(0,0,3) \\ U(0,1,0) & U(0,1,1) & U(0,1,2) & U(0,1,3) \\ U(1,0,0) & U(1,0,1) & U(1,0,2) & U(1,0,3) \\ U(1,1,0) & U(1,1,1) & U(1,1,2) & U(1,1,3) \\ U(2,0,0) & U(2,0,1) & U(2,0,2) & U(2,0,03) \\ U(2,1,0) & U(2,1,1) & U(2,1,2) & U(2,1,3) \\ U(3,0,0) & U(3,0,1) & U(3,0,2) & U(3,0,3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now by substituting this relation in equation (2.4), we get

$$u(x,y,z) = x^2y + yz^2 + xyz,$$

whichis the exact solution.

**Example 3.3.** Consider nonlinear Volterra Integral

$$u(x,y,z) = g(x,y,z) + \int_0^x\int_0^y\int_0^z u(r,s,t)dtdsdr,$$

where $(x,y,z) \in [0,1] \times [0,1] \times [0,1]$ and

$$g(x,y,z) = e^{x+y} + e^{x+z} + e^{y+z} - e^x - e^y - e^z + 1.$$

Its exact solution can be expressed as $u(x,y,z) = e^{x+y+z}$.

In the same manner in the previous example, we set

$$f(x,y,z) = \int_0^x\int_0^y\int_0^z u(r,s,t)dtdsdr,$$

then we have

$$u(x,y,z) = g(x,y,z) + f(x,y,z).$$

Taking the differential transformation of this equation, we obtain

$$U(k,h,l) = G(k,h,l) + F(k,h,l),$$

where

$$G(k,h,0) = \frac{1}{k!\,h!}\ , k = 0,1,\dots,p\ , h = 0,1,\dots,q$$

$$G(0,h,l) = \frac{1}{h!\,l!}\ , l = 0,1,\dots,n\ , h = 0,1,\dots,q$$

$$G(k,0,l) = \frac{1}{k!\,l!}\ , k = 0,1,\dots,p\ , l = 0,1,\dots,n$$

$$G(k,h,l) = 0\ , k,h,l \neq 0\ ,$$

and we also have

$$F(k,h,l) = \frac{1}{khl} U(k-1,h-1,l-1),$$

where $F(k,h,0) = F(k,0,l) = F(0,h,l) = 0$ , $k = 0,1,\dots,p\ , h = 0,1,\dots,q$ and $l = 0,1,\dots,n$. By solving the above recursive equations for $p = q = n = 2$ and $p = q = n = 3$, we obtain

$$u_{2,2,2}(x,y,z) = 1 + (x+y+z) + \frac{1}{2}(x+y+z)^2 + \frac{1}{2}(x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2) + \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2) + \frac{1}{2}(xyz^2 + xy^2z + x^2yz) + \frac{1}{4}(xy^2z^2 + x^2yz^2 + x^2y^2z) + \frac{1}{8}x^2y^2z^2,$$

$$u_{3,3,3}(x,y,z) = 1 + (x+y+z) + \frac{1}{2}(x+y+z)^2 + \frac{1}{6}(x+y+z)^3 + \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2) + \frac{1}{6}(x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3) + \frac{1}{12}(x^2y^3 + x^3y^2 + x^2z^3 + x^3z^2 + y^2z^3 + y^3z^2) + \frac{1}{36}(x^3y^3 + x^3z^3 + y^3z^3) + \frac{1}{2}(xyz^2 + xy^2z + x^2yz) + \frac{1}{4}(xy^2z^2 + x^2yz^2 + x^2y^2z) + \frac{1}{6}(xyz^3 + xy^3z + x^3yz) + \frac{1}{12}(xy^2z^3 + xy^3z^2 + x^2yz^3 + x^3yz^2 + x^2y^3z + x^3y^2z) + \frac{1}{36}(xy^3z^3 + x^3yz^3 + x^3y^3z) + \frac{1}{24}(x^2y^2z^3 + x^2y^3z^2 + x^3y^2z^2) + \frac{1}{72}(x^2y^3z^3 + x^3y^2z^3 + x^3y^3z^2) + \frac{1}{8}x^2y^2z^2 + \frac{1}{216}x^3y^3z^3,$$

which are truncated Taylor series of exact solution. Table 1 shows the absolute errors at some particular points.

Table 1

| $x$ | $y$ | $z$ | $Exact\ Solution$ | $e_{2,2,2}(x,y,z)$ | $e_{3,3,3(x,y,z)}$ |
|---|---|---|---|---|---|
| 0.1 | 0.1 | 0.1 | 1.3498588076 | $1.6261825\times 10^{-3}$ | $1.557798\times 10^{-5}$ |
| 0.01 | 0.1 | 0.1 | 1.2336780600 | $4.8175870\times 10^{-4}$ | $9.492003\times 10^{-6}$ |
| 0.01 | 0.01 | 0.1 | 1.1274968516 | $1.8474381\times 10^{-4}$ | $4.338226\times 10^{-6}$ |
| 0.01 | 0.01 | 0.01 | 1.0304545340 | $1.5113801\times 10^{-6}$ | $1.280568\times 10^{-9}$ |
| 0.001 | 0.01 | 0.01 | 1.0212220516 | $4.3803265\times 10^{-7}$ | $8.421920\times 10^{-10}$ |
| 0.001 | 0.001 | 0.01 | 1.0120722889 | $1.7775346\times 10^{-7}$ | $4.165485\times 10^{-10}$ |
| 0.001 | 0.001 | 0.001 | 1.0030045045 | $1.5043042\times 10^{-9}$ | $3.637978\times 10^{-12}$ |

According to Table 1, these results indicate that the use of time steps smaller than about 0.1, the error function approaches to zero.

# 4. Conclusion

In this study, we introduced the definition and operation of three-dimensional differential transform. Integral equations can be transformed to algebraic equations by using the differential transform and the resulting algebraic equations are called iterative equations. The overall spectra can be calculated through the initial condition in association with the iterative equations. Finally, by using this algebraic equations, we find the approximate solution of the integral equations.

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