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ON ALMOST PSEUDO CONCIRCULARLY SYMMETRIC MANIFOLDS

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Abstract

The object of the present paper is to study a type of non-flat Riemannian manifold called almost pseudo concircularly symmetric manifold. The existence of an almost pseudo concircularly symmetric manifold is also shown by two non-trivial examples.

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1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [5], who, in particular, obtained a classification of those spaces.

Let (M^n,g) (n=dim M) be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n,g). A Riemannian manifold is called locally symmetric [5] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n,g). This condition of local symmetry is equivalent to the fact that at every point

$P \in M$, the local geodesic symmetry F(P) is an isometry [18]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds have been weakended by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [7], recurrent manifolds introduced by Walker [30], conformally recurrent manifolds by Adati and Miyazawa[1], conformally symmetric Ricci-recurrent spaces by Roter[22], pseudo-Riemannian manifolds with recurrent concircular curvature tensor by Olszak and Olszak[19], semi-symmetric manifolds by Szabó[24], pseudo symmetric manifolds introduced by Chaki [6], weakly symmetric manifolds by Tamassy and Binh [25], projective symmetric manifolds by Soos [23] etc.

A non-flat Riemannian manifold (M^n,g), $(n > 2)$ is said to be a pseudo symmetric manifold [6] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho, \tag{1.1}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X), \tag{1.2}$$

for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If A=0, then the manifold reduces to a symmetric manifold in the sence of E.Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. This is to be noted that the notion of pseudo symmetric manifold studied in particular by Deszcz[15] is different from that of Chaki[6].

In 1989 Tamassy and Binh introduced the notion of weakly symmetric manifolds. A Riemannian manifold (M^n,g) $(n > 2)$ is called weakly symmetric if its curvature tensor R of type (1, 3) satisfies the condition:

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + D(Z)R(Y,X)W + E(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho, \tag{1.3}$$

where ∇ denotes the Levi-Civita connection on (M^n,g) and A, B, D, E and ρ are 1-forms and a vector field respectively, which are non-zero simultaniously. Such a manifold is denoted by $(WS)_n$. Weakly symmetric manifolds have been studied by several authors ([13],[14],[20],[21]) and many others.

In a recent paper, De and Gazi [8] introduced the notion of almost pseudo symmetric manifolds.

A Riemannian manifold (M^n, g), $(n > 2)$ is said to be almost pseudo symmetric manifold if its curvature tensor $\acute{R}$ of type (0, 4) satisfies the condition:

$$(\nabla_U \acute{R})(X,Y,Z,W)$$
$$= [A(U) + B(U)]\acute{R}(X,Y,Z,W) + A(X)\acute{R}(U,Y,Z,W) + A(Y)\acute{R}(X,U,Z,W) + A(Z)\acute{R}(X,Y,U,W) + A(W)\acute{R}(X,Y,Z,U), \quad (1.4)$$

where A, B are non-zero 1-forms defined by g(X, P) = A(X), g(X,Q) = B(X), for all vector fields X, ∇ denotes the operator of covariant differentiation with respect to the metric g, $\acute{R}$ is defined by $\acute{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, where R is the curvature tensor of type (1,3). The 1-forms A and B are called the associated 1-forms of the manifold. Such a manifold is denoted by $(APS)_n$. Here the vector fields P and Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively.
If in (1.4) B=A, then the $(APS)_n$ reduces to a $(PS)_n$. In subsequent papers ([9],[10]) De and Gazi studied almost pseudo conformally symmetric manifolds and conformally flat almost pseudo Ricci symmetric manifolds. It may be mentioned that $(PS)_n$ is a particular case of an $(APS)_n$, but $(WS)_n$ is not a particular case of an $(APS)_n$.

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \quad (1.5)$$

of the fundamental tensor g_{ij} . The transformation which preserves geodesic circles was first introduced by Yano [28]. The conformal transformation (1.5) satisfying the partial differential equation

$$\psi_{,i;j} = \phi g_{ij}, \quad (1.6)$$

changes a geodesic circle into a geodesic circle. Such a transformation is known as the concircular transformation and the geometry which deals with such transformation is called the concircular geometry [28].

A (1,3) type tensor $\tilde{C}(X,Y,Z)$ which remains invariant under concircular ransformation, for an n-dimensional Riemannian manifold M^n, is given by Yano and Kon ([27],[29])

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y, \quad (1.7)$$

where $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ (∇ being the Riemannian connection) is the Riemannian curvature tensor and r, the scalar curvature. The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain F-structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc.([4],[26],[29]). In a recent paper Ahsan and Siddiqui [2] studied the application of concircular curvature tensor in fluid spacetime.

In 1992, De and Tarafdar [11] introduced a type of non-flat Riemannian manifold (M^n, g), n > 2 whose concircular curvature tensor $\tilde{C}$ satisfies the condition

$$(\nabla_X\tilde{C})(Y,Z)W = 2A(X)\tilde{C}(Y,Z)W + A(Y)\tilde{C}(X,Z)W + A(Z)\tilde{C}(Y,X)W + A(W)\tilde{C}(Y,Z)X + g(\tilde{C}(Y,Z)W,X)\rho, \qquad (1.8)$$

where R is the curvature tensor of type (1,3), r is the scalar curvature, A is a non-zero 1-form such that g(X, ρ) = A(X) for every vector field X. Such a manifold is called a pseudo concircularly symmetric manifold and is denoted by $(P\tilde{C}S)_n$.

The object of the present paper is to study a type of non-flat Riemannian manifold (M^n,g), n > 2 whose concircular curvature tensor $\tilde{C}$ satisfies the condition:

$$(\nabla_X\tilde{C})(Y,Z)W = [A(X)+B(X)]\tilde{C}(Y,Z)W + A(Y)\tilde{C}(X,Z)W + A(Z)\tilde{C}(Y,X)W + A(W)\tilde{C}(Y,Z)X + g(\tilde{C}(Y,Z)W,X)P, \qquad (1.9)$$

where A and B are non-zero 1-forms, called associated 1-forms, defined by g(X, P) = A(X) , g(X,Q) = B(X) and P, Q are called basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively. Such a manifold shall be called an almost pseudo concircularly symmetric manifold and an n-dimensional manifold of this kind shall be denoted by $(AP\tilde{C}S)_n$. Clearly, every concircularly recurrent manifold is a $(AP\tilde{C}S)_n$. If A=B, then $(AP\tilde{C}S)_n$ reduces to a $(P\tilde{C}S)_n$.

A Riemannian manifold (M^n,g), n = dim M $\geq$ 2, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n}g, \qquad (1.10)$$

holds on M, where S and r denote the Ricci tensor and scalar curvature of (M^n, g) respectively. According to ([3],p. 432), (1.10) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in the general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([3], pp. 432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n,g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \qquad (1.11)$$

where a, b are real numbers and A is a non-zero 1-form such that

$$g(X,U) = A(X), \qquad (1.12)$$

for all vector fields X.

A non-flat Riemannian manifold (M^n,g) (n > 2) is defined to be a quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition (1.11).

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi-Einstein manifolds [16]. Also quasi-Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity [12]. So quasi-Einstein manifolds have some importance in the general theory of relativity.

Motivated by the above studies in the present paper we have studied a type of non-flat Riemannian manifold. This paper is organized as follows:
In Section 2, some geometric properties of $(AP\tilde{C}S)_n$, n > 2 have been studied under certain conditions. In the next Section it is firstly shown that if the Ricci tensor is of Codazzi type, then $(AP\tilde{C}S)_n$ reduces to a quasi-Einstein manifold under certain condition. It is also shown that $(AP\tilde{C}S)_n$ satisfies Bianchi's 2nd identity if the Ricci tensor is of Codazzi type. In Section 4, it is shown that a Ricci symmetric $(AP\tilde{C}S)_n$ is a quasi-Einstein manifold under certain condition. Finally, non-trivial examples of $(AP\tilde{C}S)_n$ have been constructed.

2. $(AP\tilde{C}S)_n$ with constant scalar curvature

Contracting (1.9) over W, we get

$$A\big(\tilde{C}(Y,Z)X\big) = 0, \qquad (2.1)$$

for all vector fields X, Y, Z.
Again contracting (1.9) over X we have

$$(\nabla_Y S)(Z,W) - (\nabla_Z S)(Y,W) - \frac{1}{n(n-1)}[g(Z,W)dr(Y) - g(Y,W)dr(Z)]$$
$$= 2A\big(\tilde{C}(Y,Z)W\big) + B\big(\tilde{C}(Y,Z)W\big) + A(Y)\Big[S(Z,W) - \frac{r}{n}g(Z,W)\Big]$$
$$- A(Z)\Big[S(Y,W) - \frac{r}{n}g(Y,W)\Big]. \qquad (2.2)$$

Using (2.1) and (2.2) we have

$$(\nabla_Y S)(Z,W) - (\nabla_Z S)(Y,W) - \frac{1}{n(n-1)}[g(Z,W)dr(Y) - g(Y,W)dr(Z)]$$
$$= B\big(\tilde{C}(Y,Z)W\big) + A(Y)\Big[S(Z,W) - \frac{r}{n}g(Z,W)\Big]$$
$$- A(Z)\Big[S(Y,W) - \frac{r}{n}g(Y,W)\Big]. \qquad (2.3)$$

Putting $Z = W = e_i$ in (2.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and i is summed for $1 \le i \le n$, we get

$$dr(Y) = \frac{2n}{n-2}\Big[\bar{B}(Y) - \frac{r}{n}A(Y) - \frac{r}{n}B(Y)\Big], \qquad (2.4)$$

where $B(LX) = \bar{B}(X)$ and L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor.

Let us suppose that the manifold under consideration is of non-zero constant scalar curvature. Then from (2.4) we get

$$\bar{B}(Y) = \frac{r}{n}[A(Y) + B(Y)]. \tag{2.5}$$

From this it follows that

$$S(Y,Q) = \frac{r}{n}[A(Y) + B(Y)]. \tag{2.6}$$

This shows that S(Y,Q) cannot be of the form λB(Y), where λ is a scalar.
Hence Q cannot be an eigen vector corresponding to any eigen value λ of S.
This leads to the following theorem:

Theorem 2.1. In an almost pseudo concircularly symmetric manifold of nonzero constant scalar curvature, Q cannot be an eigen vector corresponding to any eigen value of S.

If in particular A = B, then from (2.5) we have $\bar{B}(Y) = \frac{2r}{n}B(Y)$.
Thus we can state the following corollary:

Corollary 1. In a pseudo concircularly symmetric manifold of non-zero constant scalar curvature, $\frac{2r}{n}$ is an eigen value corresponding to the eigen vector Q.

3. $(AP\widetilde{C}S)_n$ with Codazzi type of Ricci tensor

A.Gray [17] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class $\mathcal{A}$ consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, i.e., $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$.
The class $\mathcal{B}$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e., $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$.

Suppose that the manifold under consideration satisfies Codazzi type of Ricci tensor. That is, $(\nabla_Y S)(Z,W) = (\nabla_Z S)(Y,W)$. Then it follows that r = constant.
We also suppose that $B\big(\tilde{C}(Y,Z)W\big) = 0$.
Then from (2.3) it follows that

$$A(Y)[S(Z,W) - \frac{r}{n}g(Z,W)] = A(Z)[S(Y,W) - \frac{r}{n}g(Y,W)]. \tag{3.1}$$

Putting Y = P in (3.1), we get

$$t\left[S(Z,W) - \frac{r}{n}g(Z,W)\right] = S(Z,W) - \frac{r}{n}A(Z)A(W), \qquad (3.2)$$

where $t = A(P)$ is a non-zero scalar.
Thus we have

$$S(Z,W) = cg(Z,W) + dA(Z)A(W), \qquad (3.3)$$

where $c = \frac{tr}{n(t-1)}$ and $d = \frac{r}{n(1-t)}$.
Therefore, from (3.3) it follows that the manifold is a quasi-Einstein manifold.
Hence we can state the following theorem:

Theorem 3.1. An $(AP\tilde{C}S)_n$ satisfying Codazzi type of Ricci tensor is a quasi-Einstein manifold, provided $B\big(\tilde{C}(Y,Z)W\big) = 0$.

If the scalar curvature vanishes, then from (1.7), we have $\tilde{C}(X,Y)Z = R(X,Y)Z$ and hence $\big(\nabla_W\tilde{C}\big)(X,Y)Z = (\nabla_W R)(X,Y)Z$.
This leads to the following:

Theorem 3.2. Every almost pseudo concircularly symmetric manifold of vanishing scalar curvature is an almost pseudo symmetric manifold.

Next using r=constant in (1.7), it follows that

$$\big(\nabla_X\tilde{C}\big)(Y,Z)W + \big(\nabla_Y\tilde{C}\big)(Z,X)W + \big(\nabla_Z\tilde{C}\big)(X,Y)W = 0. \qquad (3.4)$$

Thus we can state the following:

Theorem 3.3. An almost pseudo concircularly symmetric manifold whose Ricci tensor is of Codazzi type, the concircular curvature tensor satisfies Bianchi's 2nd identity.

It is well known that the concircular curvature tensor satisfies the condition

$$\tilde{C}(X,Y)Z + \tilde{C}(Y,Z)X + \tilde{C}(Z,X)Y = 0, \qquad (3.5)$$

and

$$\tilde{C}(X,Y)Z = -\tilde{C}(Y,X)Z. \qquad (3.6)$$

But, in general, the concircular curvature tensor $\tilde{C}(X,Y)Z$ does not satisfy Bianchi's 2nd identity

$$\big(\nabla_X\tilde{C}\big)(Y,Z)W + \big(\nabla_Y\tilde{C}\big)(Z,X)W + \big(\nabla_Z\tilde{C}\big)(X,Y)W = 0. \qquad (3.7)$$

We suppose that the condition (3.7) holds in the investigated almost pseudo concircularly symmetric manifold.Now using (3.5), (3.6), (3.7), we get from (1.9)

$$G(X)\tilde{C}(Y,Z)W + G(Y)\tilde{C}(Z,X)W + G(Z)\tilde{C}(X,Y)W = 0, \qquad (3.8)$$

where G(X) = B(X) – A(X) and ρ is a vector field defined by

$$g(X,\rho) = G(X). \qquad (3.9)$$

Contracting X in (3.8) we get

$$G\big(\tilde{C}(Y,Z)W\big) = 0. \qquad (3.10)$$

Now putting $X = \rho$ in (3.8) and using (3.10) we have

$$G(\rho)\tilde{C}(Y,Z)W = 0. \qquad (3.11)$$

Hence, either the manifold is concircularly flat or G(ρ) = 0. But a concircularly flat manifold is of constant curvature. Then ρ is a null vector field or the manifold is of constant curvature.
Thus we can conclude the following theorem:

Theorem 3.4. If the concircular curvature tensor of an almost pseudo concircularly symmetric manifold satisfies Bianchi's 2nd identity, then the manifold is either a manifold of constant curvature or the vector field ρ defined by (3.9) is a null vector field.

4. Ricci symmetric $(AP\widetilde{C}S)_n$

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type (0, 2) satisfies the condition $\nabla S = 0$.
Using (1.7) and (1.9) we have

$$
\begin{aligned}
(\nabla_X R)(Y,Z)W &- \frac{dr(X)}{n(n-1)}[g(Z,W)Y - g(Y,W)Z] \\
&= [A(X) + B(X)]\left[R(Y,Z)W - \frac{r}{n(n-1)}\{g(Z,W)Y - g(Y,W)Z\}\right] \\
&+ A(Y)\left[R(X,Z)W - \frac{r}{n(n-1)}\{g(Z,W)X - g(X,W)Z\}\right] \\
&+ A(Z)\left[R(Y,X)W - \frac{r}{n(n-1)}\{g(X,W)Y - g(Y,W)X\}\right] \\
&+ A(W)\left[R(Y,Z)X - \frac{r}{n(n-1)}\{g(Z,X)Y - g(Y,X)Z\}\right] \\
&+ g\big(\tilde{C}(Y,Z)W, X\big)P. \qquad (4.1)
\end{aligned}
$$

Contracting over Y, we have

$$
\begin{aligned}
(\nabla_X S)(Z,W) &- \frac{dr(X)}{n} g(Z,W) \\
&= [A(X) + B(X)]\left[S(Z,W) - \frac{r}{n} g(Z,W)\right] \\
&- \frac{r}{n(n-1)}[g(Z,W)A(X) - g(X,W)A(Z)] + A(Z)\left[S(X,W) - \frac{r}{n} g(X,W)\right] \\
&+ A(W)\left[S(Z,X) - \frac{r}{n} g(Z,X)\right] \\
&- \frac{r}{n(n-1)}[A(X)g(Z,W) - A(W)g(Z,X)]. \qquad (4.2)
\end{aligned}
$$

Let us assume that $\nabla S = 0$. Then $dr = 0$. Thus from (4.2) we have

$$
\begin{aligned}
[A(X) + B(X)]&\left[S(Z,W) - \frac{r}{n} g(Z,W)\right] - \frac{r}{n(n-1)}[g(Z,W)A(X) - g(X,W)A(Z)] \\
&+ A(Z)\left[S(X,W) - \frac{r}{n} g(X,W)\right] + A(W)\left[S(Z,X) - \frac{r}{n} g(Z,X)\right] \\
&- \frac{r}{n(n-1)}[A(X)g(Z,W) - A(W)g(Z,X)] = 0. \qquad (4.3)
\end{aligned}
$$

Putting $Z = W = e_i$ in (4.3) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \le i \le n$, we get

$$
\frac{2r}{n} A(X) = \bar{A}(X), \qquad (4.4)
$$

where $A(LX) = \bar{A}(X)$ and L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, that is, $g(LX,Y) = S(X,Y)$.

We consider the basic vector fields P and Q as A(P)=1, B(Q)=1 and g(P,Q) = 0.
Putting X = P in (4.3) and using (4.4) we get

$$S(Z,W) = ag(Z,W) + bA(Z)A(W), \tag{4.5}$$

where $a = \frac{r(n+1)}{n(n-1)}$ and $b = \frac{r(1-3n)}{n(n-1)}$.
Thus we can state the following:

Theorem 4.1. A Ricci symmetric almost pseudo concircularly symmetric manifold is a quasi-Einstein manifold provided the basic vector fields are orthonormal vector fields.

5. Examples of an $(AP\tilde{C}S)_n$

Example 5.1. Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{5.1}$$

where $q = \frac{e^{x^1}}{k^2}$ and k is non-zero constant, (i,j = 1,2,3,4.)
Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \quad \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q}, \quad R_{2332} = R_{2442} = R_{3443} = \frac{q^2}{1+2q}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are:

$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{q}{1+2q}.$$

.

It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is $\frac{6q(1+q)}{(1+2q)^3}$, which is non-vanishing and non-constant. The non-vanishing components of concircular curvature and their covariant derivatives are:

$$\tilde{C}_{1221} = \tilde{C}_{1331} = \tilde{C}_{1441} = \frac{q-q^2}{2(1+2q)}, \quad \tilde{C}_{2332} = \tilde{C}_{2442} = \tilde{C}_{3443} = -\frac{q-q^2}{2(1+2q)},$$

$$\tilde{C}_{1221,1} = \tilde{C}_{1331,1} = \tilde{C}_{1441,1} = \frac{q-2q^2-2q^3}{2(1+2q)^2},$$

$$\tilde{C}_{2332,1} = \tilde{C}_{2442,1} = \tilde{C}_{3443,1} = -\frac{q-2q^2-2q^3}{2(1+2q)^2}.$$

Let us choose the associated 1-forms as follows:

$A_i(x) = \frac{1}{3}\frac{1}{1+2q}$ for i=1
= 0 otherwise,

$B_i(x) = \frac{q}{q-1}$ for i=1
= 0 otherwise,

at any point $x \in \mathbb{R}^4$.

Now equation (1.9) reduces to

$$\tilde{C}_{1221,1}=(3A_1+B_1)\tilde{C}_{1221}, \quad (5.2)$$
$$\tilde{C}_{1331,1}=(3A_1+B_1)\tilde{C}_{1331}, \quad (5.3)$$
$$\tilde{C}_{1441,1}=(3A_1+B_1)\tilde{C}_{1441}, \quad (5.4)$$
$$\tilde{C}_{2332,1}=(3A_1+B_1)\tilde{C}_{2332}, \quad (5.5)$$
$$\tilde{C}_{2442,1}=(3A_1+B_1)\tilde{C}_{2442}, \quad (5.6)$$
$$\tilde{C}_{3443,1}=(3A_1+B_1)\tilde{C}_{3443}. \quad (5.7)$$

R.H.S. of (5.2) = $(3A_1 + B_1)\tilde{C}1221$
$=(\frac{1}{1+2q} + \frac{q}{q-1})\frac{q-q^2}{2(1+2q)}$
$=\frac{q-2q^2-2q^3}{2(1+2q)^2}$
$=\tilde{C}1221,1$
= L.H.S of (5.2).

By similar argument it can be shown that (5.3), (5.4), (5.5), (5.6) and (5.7) are also true. So, $(\mathbb{R}^4, g)$ is an almost pseudo concircularly symmetric manifold whose scalar curvature is non-zero and non-constant.
We shall now show that this manifold $(\mathbb{R}^4, g)$ is not an almost pseudo symmetric manifold. Now equation (1.4) reduces to

$$R_{1221,1} = (3A_1 + B_1)R_{1221}, \quad (5.8)$$
$$R_{1331,1} = (3A_1 + B_1)R_{1331}, \quad (5.9)$$
$$R_{1441,1} = (3A_1 + B_1)R_{1441}, \quad (5.10)$$
$$R_{2332,1} = (3A_1 + B_1)R_{2332}, \quad (5.11)$$
$$R_{2442,1} = (3A_1 + B_1)R_{2442}, \quad (5.12)$$
$$R_{3443,1} = (3A_1 + B_1)R_{3443}. \quad (5.13)$$

The above system of equations is equivalent to the following:

$$3A_1 + B_1 = \frac{1}{1+2q},$$

$$3A_1 + B_1 = \frac{2(1+q)}{1+2q}.$$

From the above it follows that there does not exist solution for A1 and B1. So the manifold $(\mathbb{R}^4, g)$ is not an almost pseudo symmetric manifold.
Thus we can state the following theorem:

Theorem 5.1. Let (M^4,g)) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $q = \frac{e^{x^1}}{k^2}$ and k is non-zero constant, $(i, j = 1,2,3,4)$. Then (M^4,g) is an almost pseudo concircularly symmetric manifold with non-zero non-constant scalar curvature which is not an almost pseudo symmetric manifold.

Example 5.2. Let M^4 be an open subset of R^4 endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2, \qquad (i, j = 1,2,3,4),$$

where $f = a_0 + a_1x^3 + e^{x^1}\left[\frac{(x3)^2}{2} - \frac{(x3)^3}{6} + \frac{(x3)^4}{12} - \ldots. \quad + \frac{(-1)^{n-1}(x3)^{n+1}}{n(n+1)} + .. \right]$, a_0, a_1 are non-constant functions of x^1 only and $-1 < x^3 \leq 1$. Then the only non-vanishing components of the Christoffel symbol and curvature tensor are:

$$\Gamma_{11}^3 = -\frac{1}{2}f_{.3}\,, \qquad R_{1331} = \frac{e^{x^1}}{2(1+x^3)}\,,$$

where '.' denotes the partial differentiation with respect to the coordinates and the components obtained by the symmetry properties. The non-vanishing component of the Ricci tensor is $R_{11} = \frac{e^{x^1}}{2(1+x^3)}$ and the scalar curvature r is 0. The only non-vanishing component of concircular curvature is given by

$$\tilde{C}1331 = \frac{e^{x^1}}{2(1+x^3)}.$$

Since r = 0, (M^4,g) is an almost pseudo concircularly symmetric manifold as well as an almost pseudo symmetric manifold which justifies Theorem 3.2., provided that n = 4.

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