Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 17 (2017), 148–157
Research Article

Online: ISSN 2008-949x

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Another class of warped product CR-submanifolds in Kenmotsu manifolds

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Abstract

Recently, Arslan et al. [K. Arslan, R. Ezentas, I. Mihai, C. Murathan, J. Korean Math. Soc., **42** (2005), 1101–1110] studied contact CR-warped product submanifolds of the form $M_T \times_f M_\perp$ of a Kenmotsu manifold $\widetilde{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\widetilde{M}$, respectively. In this paper, we study the warped product submanifolds by reversing these two factors, i.e., the warped products of the form $M_\perp \times_f M_T$ which have not been considered in earlier studies. On the existence of such warped products, a characterization is given. A sharp estimation for the squared norm of the second fundamental form is obtained, and in the statement of inequality, the equality case is considered. Finally, we provide two examples of non-trivial warped product submanifolds.

Keywords: Warped products, contact CR-submanifolds, mixed totally geodesic, contact CR-warped products, Kenmotsu manifolds.

2010 MSC: 53C25, 53C40, 53C42, 53B25.

1. Introduction

To construct the examples of manifolds with negative curvature, the warped product manifolds were studied by Bishop and O'Neill in [4]. They defined these manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider a product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y),$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be trivial or simply a Riemannian product manifold, if the warping function f is constant. Let X be a vector field tangent to M_1 and Z be an another vector field on M_2, then from [4, Lemma 7.3], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z, \tag{1.1}$$

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doi:10.22436/jmcs.017.01.13

Received 2016-12-12

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ be a warped product manifold then M_1 is a totally geodesic submanifold of M and M_2 is a totally umbilical submanifold of M [4, 5].

In the beginning of this century, Chen introduced the notion of warped product CR-submanifolds of Kaehler manifolds [5]. After that, many researchers extended this idea for other structures on a Riemannian manifold (some of them are cited here [8, 14]). For the survey on warped product submanifolds, we refer to [6, 7].

On the other hand, several results on the warped product submanifolds of Kenmotsu manifolds appeared in [1–3, 11–13, 17, 18]. In this paper, we study some geometric properties of warped product submanifolds of the form $M_\perp \times_f M_T$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of a Kenmotsu manifold $\widetilde{M}$, respectively. A characterization is given on the existence of such type of warped products. Also, we establish a relationship between the squared norm of second fundamental form $\|\sigma\|^2$ and the warping function f. The equality case in the statements of the inequality is considered. Furthermore, we construct non-trivial examples of warped product contact CR-submanifolds.

2. Preliminaries

Tanno [15] has classified the connected almost contact metric manifolds into 3 classes whose automorphism groups have maximum dimensions:

(a) Homogeneous normal contact Riemannian manifolds with constant φ holomorphic sectional curvature;
(b) global Riemannian product of a line or a circle and a Kaehlerian space form;
(c) warped product spaces $L \times_f F$, where L is a line and F a Kaehlerian manifold.

Kenmotsu [10] studied the class (c) and characterized it by tensor equations. Later such manifolds were called Kenmotsu manifolds.

A $(2n+1)$-dimensional Riemannian manifold $(\widetilde{M}, g)$ is said to be a Kenmotsu manifold, if it admits an endomorphism φ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η satisfying the following conditions:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.1}$$

$$(\widetilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \widetilde{\nabla}_X \xi = X - \eta(X)\xi, \tag{2.2}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ is the Riemannian connection with respect to g.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively, then the Gauss-Weingarten formulas are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{2.3}$$

for any vector field X, Y tangent to M and V normal to M, where σ and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\widetilde{M}$. They are related as $g(\sigma(X, Y), V) = g(A_V X, Y)$ where g denotes the Riemannian metric on $\widetilde{M}$ as well as the one induced on M.

Let $\widetilde{M}$ be a Kenmotsu manifold and M an m-dimensional submanifold tangent to ξ. For any X tangent to M, we put

$$\varphi X = PX + FX,$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX. Then P is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM.

We denote by H, the mean curvature vector, i.e.,

$$H(p) = \sum_{i=1}^{m} \sigma(e_i, e_i),$$

where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the tangent space T_pM, for any $p \in M$.

Also, we set

$$\sigma_{ij}^{r} = g(\sigma(e_i, e_j), e_r), \quad i, j = 1, \cdots, m, \quad r = m+1, \cdots, 2n+1,$$

and

$$\|\sigma\|^2 = \sum_{i,j=1}^{m} g(\sigma(e_i, e_j), \sigma(e_i, e_j)).$$

For a differentiable function f on an n-dimensional manifold M, the gradient $\vec{\nabla} f$ of f is defined as $g(\vec{\nabla} f, X) = Xf$, for any X tangent to M. As a consequence, we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^{n} (e_i(f))^2,$$

for an orthonormal frame $\{e_1 \cdots, e_n\}$ on M.

By the analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered.

A submanifold M tangent to ξ is said to be invariant (resp. anti-invariant), if $\varphi(T_pM) \subset T_pM$, for all $p \in M$ (resp. $\varphi(T_pM) \subset T_p^{\perp}M$, for all $p \in M$).

A submanifold M tangent to ξ is said to be a contact CR-submanifold, if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^{\perp} : p \to \mathcal{D}_p^{\perp}$, for all $p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\sim\rangle$, where $\langle\xi\rangle$ is a 1-dimensional distribution spanned by ξ;
(ii) $\mathcal{D}$ is invariant by φ, i.e., $\varphi\mathcal{D} = \mathcal{D}$;
(iii) $\mathcal{D}^{\perp}$ is anti-invariant by φ, i.e., $\varphi\mathcal{D}^{\perp} \subseteq T^{\perp}M$.

Invariant and anti-invariant submanifolds are the special cases of contact CR-submanifolds. If we denote the dimensions of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by d_1 and d_2 respectively, then M is invariant (resp. anti-invariant), if $d_2 = 0$ (resp. $d_1 = 0$).

For the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$, we have the following results for later use.

Lemma 2.1 ([12]). *The φ anti-invariant distribution $\mathcal{D}^{\perp} \oplus \langle\xi\rangle$ on a contact CR-submanifold of a Kenmotsu manifold $\widetilde{M}$ is always integrable.*

Also, we can prove the following result for the integrability of the distribution $\mathcal{D}$.

Lemma 2.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\widetilde{M}$. Then the φ invariant distribution $\mathcal{D}$ is integrable, if and only if*

$$g(\nabla_Y X, Z) = g(\sigma(X, \varphi Y), \varphi Z) - \eta(Z) g(X, Y),$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$.

Proof. For any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, we have

$$\begin{aligned} g([X,Y],Z) &= g(\varphi\widetilde{\nabla}_X Y, \varphi Z) + \eta(Z) g(\widetilde{\nabla}_X Y, \xi) - g(\widetilde{\nabla}_Y X, Z) \\ &= g(\widetilde{\nabla}_X \varphi Y, \varphi Z) - g((\widetilde{\nabla}_X \varphi)Y, \varphi Z) - \eta(Z) g(Y, \widetilde{\nabla}_X \xi)) - g(\widetilde{\nabla}_Y X, Z). \end{aligned}$$

Using (2.2) and (2.3), we get

$$g([X,Y],Z) = g(\sigma(X,\varphi Y), \varphi Z) - \eta(Z) g(X,Y) - g(\widetilde{\nabla}_Y X, Z).$$

Thus the result follows from the above equation. □

Lemma 2.3. *On a contact CR-submanifold M of a Kenmotsu manifold $\widetilde{M}$. The following statement holds*

$$g([X,Y],Z) = g(\sigma(X,\varphi Y), \varphi Z) - g(\sigma(\varphi X, Y), \varphi Z),$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$.

Proof. For any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, \varphi Z) + \eta(Z) g(\widetilde{\nabla}_X Y, \xi).$$

Using the covariant derivative property of φ and the Kenmotsu structure equation (2.2), we derive

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X \varphi Y, \varphi Z) - \eta(Z) g(Y, \widetilde{\nabla}_X \xi).$$

Then from (2.3) we get

$$g(\nabla_X Y, Z) = g(\sigma(X,\varphi Y), \varphi Z) - \eta(Z) g(X,Y). \tag{2.4}$$

Interchanging X and Y in (2.4), we obtain

$$g(\nabla_Y X, Z) = g(\sigma(Y,\varphi X), \varphi Z) - \eta(Z) g(X,Y). \tag{2.5}$$

Then from (2.4) and (2.5), we get the desired result. □

Now, we have the following consequence of the above lemma.

Corollary 2.4. *On a contact CR-submanifold M of a Kenmotsu manifold $\widetilde{M}$. The invariant distribution $\mathcal{D}$ is integrable, if and only if*

$$\textit{either} \ \ \sigma(X,\varphi Y) = \sigma(\varphi X, Y) \ \ \textit{or} \ \ \sigma(\varphi X, Y) \in \nu,$$

for any $X, Y \in \mathcal{D}$.

3. Contact CR-warped products

In [1], Arslan et al. studied warped product contact CR-submanifolds of the form $M_T \times_f M_\perp$, called contact CR-warped products of a Kenmotsu manifold $\widetilde{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\widetilde{M}$, respectively. They establish an inequality for such type of warped products. In this paper, we study another type of warped products by reversing two factors which have not been considered in [1]. First, we prove the following results for later use.

Lemma 3.1. *Let $M = M_\perp \times {}_f M_T$ be a warped product submanifold of a Kenmotsu manifold $\widetilde{M}$ such that $\xi \in TM_\perp$, then we have*

(i) $g(\sigma(X,Y),\varphi Z)=\{(Z\ln f)-\eta(Z)\}g(X,\varphi Y)$;
(ii) $g(\sigma(X,Z),\varphi W)=g(\sigma(X,W),\varphi Z)$;
(iii) $g(\sigma(Z,W),\varphi W')=g(\sigma(Z,W'),\varphi W)$;

for any $X,Y\in TM_T$ *and* $Z,W,W'\in TM_\perp$.

Proof. From (2.3) and (2.2), we have

$$g(\sigma(X,Y),\varphi Z)=g(\widetilde{\nabla}_XY,\varphi Z)=g((\widetilde{\nabla}_X\varphi)Y,Z)-g(\widetilde{\nabla}_X\varphi Y,Z)=\eta(Z)g(\varphi X,Y)+g(\varphi Y,\widetilde{\nabla}_XZ),$$

for any $X,Y\in TM_T$ and $Z\in TM_\perp$. Then the first part follows from the above relation by using (1.1). Also the second and third parts of the lemma can be derived by using (2.2), (2.3) and the orthogonality of vector fields. □

If we interchange X by φX and Y by φY in the first part of Lemma 3.1, then we get the following relations

$$g(\sigma(\varphi X,Y),\varphi Z)=\{(Z\ln f)-\eta(Z)\}g(X,Y), \tag{3.1}$$

$$g(\sigma(X,\varphi Y),\varphi Z)=\{\eta(Z)-(Z\ln f)\}g(X,Y), \tag{3.2}$$

and

$$g(\sigma(\varphi X,\varphi Y),\varphi Z)=\{(Z\ln f)-\eta(Z)\}g(X,\varphi Y). \tag{3.3}$$

Corollary 3.2. *On a contact CR-warped product submanifold* $M=M_\perp\times{}_fM_T$ *of a Kenmotsu manifold* $\widetilde{M}$, *we have*

$$g(\sigma(\varphi X,Y),\varphi Z)=-g(\sigma(X,\varphi Y),\varphi Z),$$

and

$$g(\sigma(\varphi X,\varphi Y),\varphi Z)=g(\sigma(X,Y),\varphi Z),$$

for any $X,Y\in TM_T$ *and* $Z\in TM_\perp$.

Proof. The first part follows from (3.1) and (3.2) and the second part follows from Lemma 3.1 (i) and (3.3). □

Theorem 3.3. *Let* M *be a contact CR-submanifold of a Kenmotsu manifold* $\widetilde{M}$ *such that* ξ *is orthogonal to the invariant distribution* $\mathcal{D}$. *Then* M *is a locally warped product submanifold, if and only if*

$$A_{\varphi Z}X=-\{(Z\mu)-\eta(Z)\}\varphi X, \tag{3.4}$$

for any $X\in\mathcal{D}$ *and* $Z\in\mathcal{D}^\perp\oplus\langle\xi\rangle$ *for some smooth function* μ *on* M *such that* $Y(\mu)=0$, *for any* $Y\in\mathcal{D}$.

Proof. If M is a contact CR-warped product submanifold, then for any $X\in TM_T$ and $Z,W\in TM_\perp$, we have

$$g(A_{\varphi Z}X,W)=g(\sigma(X,W),\varphi Z)=g(\widetilde{\nabla}_WX,\varphi Z)=-g(\varphi\widetilde{\nabla}_WX,Z).$$

Using a covariant derivative property of φ and (2.2), we find

$$g(A_{\varphi Z}X,W)=-g(\widetilde{\nabla}_W\varphi X,Z).$$

Then from (2.3) and (1.1), we get $g(A_{\varphi Z}X, W) = 0$, i.e., $A_{\varphi Z}X$ has no components in $TM_\perp$. Therefore, from Lemma 3.1, the relation (3.4) holds.

Conversely, if M is a contact CR-submanifold with the invariant and anti-invariant distributions $\mathcal{D}$ and $\mathcal{D}^\perp \oplus \langle\xi\rangle$ such that the given condition (3.4) holds, then, by Lemma 2.1, $\mathcal{D}^\perp \oplus \langle\xi\rangle$ is always integrable and for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp \oplus \langle\xi\rangle$, we have

$$g(\nabla_Z W, \varphi X) = g((\widetilde{\nabla}_Z \varphi)W, X) - g(\widetilde{\nabla}_Z \varphi W, X).$$

Using (2.2) and (2.3), we get

$$g(\nabla_Z W, \varphi X) = g(\varphi W, \widetilde{\nabla}_Z X) = g(\sigma(X, Z), \varphi W) = g(A_{\varphi W}X, Z).$$

Then from (3.4), we get $g(\nabla_Z W, \varphi X) = 0$, which means that the leaves of the distribution $\mathcal{D}^\perp \oplus \langle\xi\rangle$ are totally geodesic in M. On the other hand, on a contact CR-submanifold from Lemma 2.3, we have

$$g([X, Y], Z) = g(A_{\varphi Z}X, \varphi Y) - g(A_{\varphi Z}Y, \varphi X),$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp \oplus \langle\xi\rangle$. Using (3.4), (2.1) and the fact that the structure vector field ξ is orthogonal to $\mathcal{D}$, we get $g([X, Y], Z) = 0$, which means that $\mathcal{D}$ is integrable. Let us consider a leaf M_T of $\mathcal{D}$ in M and let $\sigma^\#$ be the second fundamental from of M_T in M, then we have

$$g(\sigma^\#(X, Y), Z) = g(\nabla_Y X, Z) = g(\varphi\widetilde{\nabla}_Y X, \varphi Z) + \eta(Z)g(\widetilde{\nabla}_Y X, \xi),$$

which on using the covariant derivative property of φ and the orthogonality of vector fields, we have

$$g(\sigma^\#(X, Y), Z) = g(\widetilde{\nabla}_Y \varphi X, \varphi Z) - g((\widetilde{\nabla}_Y \varphi)X, \varphi Z) - \eta(Z)g(\widetilde{\nabla}_Y \xi, X).$$

Using (2.2), we get

$$g(\sigma^\#(X, Y), Z) = -g(\varphi X, \widetilde{\nabla}_Y \varphi Z) - \eta(Z)g(X, Y).$$

Then from (2.3), we obtain

$$g(\sigma^\#(X, Y), Z) = g(\varphi X, A_{\varphi Z}Y) - \eta(Z)g(X, Y).$$

From (3.4), we find

$$g(\sigma^\#(X, Y), Z) = -(Z\mu)g(X, Y),$$

or equivalently

$$\sigma^\#(X, Y) = -\vec{\nabla}\mu g(X, Y),$$

where $\vec{\nabla}\mu$ is gradient of the function μ, which means that M_T is totally umbilical in M with the mean curvature $H^\# = -\vec{\nabla}\mu$. Also, it is easy to prove that $H^\#$ is parallel corresponding to the normal connection $D^\#$ of M_T in M (see [16]). Thus, M_T is an extrinsic sphere in M. Hence, by a result of Hiepko [9] we conclude that M is a warped product submanifold, which proves the theorem completely. □

Now, we set the following orthonormal frame for the warped product submanifold $M = M_\perp \times_f M_T$ of a $(2n+1)$-dimensional Kenmotsu manifold $\widetilde{M}$ with the fiber M_T of dimension $2p$ and the base $M_\perp$ of dimension $q+1$ such that ξ is tangent to $M_\perp$. Let us consider the tangent spaces of M_T and $M_\perp$ by $\mathcal{D}$ and $\mathcal{D}^\perp \oplus \langle\xi\rangle$, respectively. We set the orthonormal frame fields of $\mathcal{D}$ and $\mathcal{D}^\perp \oplus \langle\xi\rangle$, respectively as $\{e_1, \cdots, e_p, e_{p+1} = \varphi e_1, \cdots, e_{2p} = \varphi e_p\}$ and $\{e_{2p+1} = e_1^*, \cdots, e_{2p+q} = e_q^*, e_{2p+q+1} = e_{q+1}^* = \xi\}$.

Then the orthonormal frame fields of the normal subbundles of $\varphi\mathcal{D}^\perp$ and ν, respectively are $\{e_{m+1} = \varphi e_1^*, \cdots, e_{m+q} = \varphi e_q^*\}$ and $\{e_{m+q+1}, \cdots, e_{2n+1}\}$.

A warped product submanifold $M = M_1 \times_f M_2$ of a Riemannian manifold $\widetilde{M}$ is said to be mixed totally geodesic, if $\sigma(X, Z) = 0$, for any $X \in TM_1$ and $Z \in TM_2$, where M_1 and M_2 are any Riemannian submanifolds of $\widetilde{M}$.

Now, we establish the following inequality for the squared norm of the second fundamental form of $M = M_\perp \times_f M_T$ of a Kenmotsu manifold $\widetilde{M}$.

Theorem 3.4. *Let $M = M_\perp \times_f M_T$ be a warped product submanifold of a Kenmotsu manifold $\widetilde{M}$ such that $\xi \in TM_\perp$, where $M_\perp$ is an anti-invariant submanifold and M_T is an invariant submanifold of $\widetilde{M}$. Then:*

(i) *The squared norm of the second fundamental form σ of M satisfies*

$$\|\sigma\|^2 \geqslant 2p[\|\vec{\nabla} \ln f\|^2 - 1],$$

where $2p = \dim M_T$ and $\vec{\nabla} \ln f$ is gradient of the function $\ln f$ along $M_\perp$.

(ii) *If equality sign in (i) holds identically, then $M_\perp$ and M_T are totally geodesic and totally umbilical submanifolds of $\widetilde{M}$, respectively. Moreover, M is a mixed totally geodesic submanifold of $\widetilde{M}$.*

Proof. By definition of σ, we have

$$\|\sigma\|^2 = \sum_{i,j=1}^{m} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(\sigma(e_i, e_j), e_r)^2.$$

Then by using the above mentioned frame, we derive

$$\begin{aligned} \|\sigma\|^2 = &\sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{2p} g(\sigma(e_i, e_j), e_r)^2 + 2 \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{2p} \sum_{j=1}^{q+1} g(\sigma(e_i, e_j^*), e_r)^2 \\ &+ \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{q+1} g(\sigma(e_i^*, e_j^*), e_r)^2. \end{aligned} \tag{3.5}$$

By leaving second and third positive terms in the right hand side of (3.5), the above expression will be

$$\|\sigma\|^2 \geqslant \sum_{r=1}^{q} \sum_{i,j=1}^{2p} g(\sigma(e_i, e_j), \varphi e_r^*)^2 + \sum_{r=m+q+1}^{2n+1} \sum_{i,j=1}^{2p} g(\sigma(e_i, e_j), e_r)^2. \tag{3.6}$$

The second term in the right hand side of the above relation has the ν-components only, therefore we will also leave this term and thus for the frame of $\mathcal{D}$, the inequality (3.6) reduces to

$$\begin{aligned} \|\sigma\|^2 \geqslant &\sum_{r=1}^{q} \sum_{i,j=1}^{p} g(\sigma(e_i, \varphi e_j), \varphi e_r^*)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{p} g(\sigma(\varphi e_i, e_j), \varphi e_r^*)^2 \\ &+ \sum_{r=1}^{q} \sum_{i,j=1}^{p} g(\sigma(\varphi e_i, \varphi e_j), \varphi e_r^*)^2 + \sum_{r=1}^{q} \sum_{i,j=1}^{p} g(\sigma(e_i, e_j), \varphi e_r^*)^2. \end{aligned}$$

Using Lemma 3.1 (i) and the relations (3.1), (3.2), (3.3), the third and the last terms of right hand side are identically zero. Thus, we derive

$$\|\sigma\|^2 \geqslant p \sum_{r=1}^{q} [\eta(e_r^*) - e_r^* \ln f]^2 + p \sum_{r=1}^{q} [e_r^* \ln f - \eta(e_r^*)]^2$$

$$= 2p \sum_{r=1}^{q+1} [\eta(e_r^*) - e_r^* \ln f]^2 - 2p[\eta(e_{q+1}^*) - (e_{q+1}^* \ln f)]^2.$$

Since $e_{q+1}^* = \xi$ and for a warped product Riemannian submanifold M of a Kenmotsu manifold $\widetilde{M}$, $(\xi \ln f) = 1$ [1, 17]. Then the above inequality will be

$$\begin{aligned}\|\sigma\|^2 &\geqslant 2p[\eta(e_{q+1}^*)^2 + \sum_{r=1}^{q+1}(e_r^* \ln f)^2 - 2\sum_{r=1}^{q+1}(e_r^* \ln f)\eta(e_r^*)] \\ &= 2p[\|\vec{\nabla} \ln f\|^2 - 1],\end{aligned}$$

which is inequality (i). If the equality holds in (i), then from the remaining terms in (3.5), we get

$$\sigma(\mathcal{D}^\perp, \mathcal{D}) = 0, \quad \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{3.7}$$

Also, from the remaining second term in the right hand side of (3.6), we find

$$\sigma(\mathcal{D}, \mathcal{D}) \perp \nu \ \Rightarrow \ \sigma(\mathcal{D}, \mathcal{D}) \in \varphi\mathcal{D}^\perp. \tag{3.8}$$

The second condition of (3.7) implies that $M_\perp$ is a totally geodesic submanifold of $\widetilde{M}$ due to $M_\perp$ being totally geodesic in M [4, 5]. On the other hand, (3.8) implies that M_T is totally umbilical in $\widetilde{M}$ with the fact that M_T is totally umbilical in M [4, 5]. Moreover, all conditions of (3.7) and (3.8) imply that M is a mixed totally geodesic submanifold of $\widetilde{M}$. Hence the proof is complete. □

In [14], Olteanu established the following estimation for the squared norm of the second fundamental form for contact CR-doubly warped products in Kenmotsu manifolds.

Theorem 3.5 ([14]). *Let $\widetilde{M}$ be a $(2m+1)$-dimensional Kenmotsu manifold and $M = {}_{f_2}M_1 \times_{f_1} M_2$ an n-dimensional contact CR-doubly warped product submanifold, such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 is a β-dimensional anti-invariant submanifold of $\widetilde{M}$. Then:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|\sigma\|^2 \geqslant 2\beta\left(\|\vec{\nabla}(\ln f_1)\|^2 - 1\right), \tag{3.9}$$

where $\vec{\nabla}(\ln f_1)$ is the gradient of $\ln f_1$.

(ii) *If the equality sign of (3.9) holds identically, then M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of M . Moreover, M is a minimal submanifold of $\widetilde{M}$.*

Now, we give the following examples of non-trivial warped product contact CR-submanifolds of the forms $M_\perp \times_f M_T$ and $M_\perp \times_f M_T$ and in both the cases the structure vector field ξ is tangent to the base manifold.

Example 3.6. Consider the Kenmotsu manifold $\widetilde{M} = \mathbb{R} \times_f \mathbb{C}^4$ with the structure (φ, ξ, η, g) is given by

$$\varphi\left\{\sum_{i=1}^{4}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial t}\right\} = \sum_{i=1}^{4}\left(-Y_i\frac{\partial}{\partial x^i} + X_i\frac{\partial}{\partial y^i}\right),$$

$$\xi = 2e^{-t}\left(\frac{\partial}{\partial t}\right), \quad \eta = \frac{1}{2}e^t dt, \quad \text{and} \quad g = \eta \otimes \eta + \frac{e^{2t}}{4}\sum_{i=1}^{4}\left(dx^i \otimes dx^i + dy^i \otimes dy^i\right).$$

Consider the submanifold M of $\widetilde{M}$ defined by

$$x(u,v,w,s,t) = 2(e^{-t}u,\ 0,\ w,\ 0,\ 0,\ e^{-t}v,\ s,\ 0,\ t).$$

Then M is a contact CR-warped product submanifold with the integrable invariant and anti-invariant distributions $\mathcal{D} = \{e_3, e_4,\}$ and $\mathcal{D}^{\perp} = \{e_1, e_2, e_5\}$ such that

$$e_1 = \frac{2}{e^t}\left(\frac{\partial}{\partial x^1}\right),\quad e_2 = \frac{2}{e^t}\left(\frac{\partial}{\partial y^2}\right),\quad e_3 = 2\left(\frac{\partial}{\partial x^3}\right),$$

$$e_4 = 2\left(\frac{\partial}{\partial y^3}\right),\quad e_5 = 2\left(\frac{\partial}{\partial t}\right) = e^t\xi.$$

Consider, the integral manifolds corresponding to the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by M_T and $M_{\perp}$, respectively. Then their corresponding Riemannian metrics are respectively given by $g_{M_T} = e^{2t}\left((dw)^2 + (ds)^2\right)$ and $g_{M_{\perp}} = dt^2 + du^2 + dv^2$. Thus $M = M_{\perp} \times_f M_T$ is a warped product submanifold isometrically immersed in $\widetilde{M}$ with metric $g = g_{M_{\perp}} + e^{2t} g_{M_T}$ and warping function $f = e^t$.

Example 3.7. Consider a submanifold of $\mathbb{R}^7$ with the Cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\quad \varphi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},\quad \varphi\left(\frac{\partial}{\partial t}\right) = 0,\quad 1 \leqslant i,j \leqslant 3.$$

Then for some smooth functions $\lambda_i,\ \nu_j$ and μ on $\mathbb{R}^7$, for any $i,j = 1,2,3$, consider a vector field $X = \lambda_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \mu \frac{\partial}{\partial t} \in \mathbb{R}^7$, we have

$$g(X,X) = \lambda_i^2 + \nu_j^2 + \mu^2,\quad g(\varphi X, \varphi X) = \lambda_i^2 + \nu_j^2,$$

and

$$\varphi^2(X) = -\lambda_i \frac{\partial}{\partial x_i} - \nu_j \frac{\partial}{\partial y_j} = -X + \eta(X)\xi,$$

for any $i,j = 1,2,3$. It is clear that $g(\varphi X, \varphi X) = g(X,X) - \eta^2(X)$. Thus, (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Let us consider a submanifold M of $\mathbb{R}^7$ defined by the immersion χ as follows

$$\chi(u,v,w,t) = (u\cos w, v\cos w, u\sin w, v\sin w, 0, 0, t).$$

Then the tangent bundle TM is spanned by the following orthogonal vector fields

$$e_1 = \cos w\frac{\partial}{\partial x_1} + \sin w\frac{\partial}{\partial x_2},\quad e_2 = \cos w\frac{\partial}{\partial y_1} + \sin w\frac{\partial}{\partial y_2},$$

$$e_3 = -u\sin w\frac{\partial}{\partial x_1} - v\sin w\frac{\partial}{\partial y_1} + u\cos w\frac{\partial}{\partial x_2} + v\cos w\frac{\partial}{\partial y_2},\quad e_4 = \frac{\partial}{\partial t}.$$

Then, we find

$$\varphi e_1 = \cos w\frac{\partial}{\partial y_1} + \sin w\frac{\partial}{\partial y_2},\quad \varphi e_2 = -\cos w\frac{\partial}{\partial x_1} - \sin w\frac{\partial}{\partial x_2},$$

$$\varphi e_3 = -u\sin w\frac{\partial}{\partial y_1} + v\sin w\frac{\partial}{\partial x_1} + u\cos w\frac{\partial}{\partial y_2} - v\cos w\frac{\partial}{\partial x_2},\quad \varphi e_4 = 0.$$

Thus M is a contact CR-submanifold with invariant distribution $\mathcal{D} = span\{e_1, e_2\}$ and anti-invariant distribution $\mathcal{D}^\perp = span\{e_3\}$ such that $\xi = e_4$ tangent to $\mathcal{D}$. Also, it is easy to see that both the distributions $\mathcal{D} \oplus <\xi>$ and $\mathcal{D}$ are integrable. If we denote the integral manifolds of $\mathcal{D}$ and $\mathcal{D}^\perp \oplus <\xi>$ by M_T and $M_\perp$ respectively, then the metric tensor g_M of M is given by

$$g_M = du^2 + dv^2 + dt^2 + (u^2 + v^2)dw^2,$$

where $g_{M_T} = du^2 + dv^2 + dt^2$ is the metric tensor of M_T and $g_{M_\perp} = (u^2 + v^2)dw^2$ is the metric tensor of $M_\perp$. Thus M is a warped product contact CR-submanifold $M = M_T \times_f M_\perp$ with the warping function $f = \sqrt{u^2 + v^2}$.

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