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On a Pseudo Projective $\phi-$ Recurrent Sasakian Manifolds

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Abstract

The object of the present paper is to study the pseudo projective ϕ –recurrent Sasakian manifolds.

Keywords: Pseudo projective $\phi-$ symmetric manifold, pseudo projective $\phi-$ recurrent manifold, Einstein manifold.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. As a weaker version of local symmetry, in 1977, Takahashi [9] introduced the notion of locally $\phi-$symmetric Sasakian manifold and obtained their several interesting results. The properties of pseudo projective curvature tensor is studied by many geometers [17], [18], [19], [22] and obtained their some interesting results.

In this paper we shown that pseudo projective $\phi-$ recurrent Sasakian manifold is an Einstein manifold and in a pseudo projective $\phi-$ recurrent Sasakian manifold, the characteristic vector field ξ and the vector field ρ associated to the $1-$ form A are co-directional. Finally, we proved that a three dimensional locally pseudo – projective $\phi-$ recurrent Sasakian manifold is of constant curvature.

2. Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost contact Riemannian manifold, where ϕ is a $(1,1)$ tensor field, ξ is the structure vector field, η is a $1-$ form and g is the Riemannian metric.It is well known that the structure (ϕ,ξ,η,g) satisfy

$$\phi^2X=-X+\eta(X)\xi, \tag{1}$$

$$(a)\ \eta(\xi)=1, (b)\ g(X,\xi)=\eta(X), (c)\ \eta(\phi X)=0, (d)\ \phi\xi=0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{3}$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{4}$$
$$\nabla_X \xi = -\varphi X, \tag{5}$$
$$(\nabla_X \eta)(Y) = g(X, \phi Y), \tag{6}$$

for all vector fields X, Y, Z, where ∇ denotes the operator of covariant differentiation with respect to g, then $M^{2n+1}(\phi, \xi, \eta, g)$ is called a Sasakian manifold [1].

Sasakian manifolds have been studied by many authors such as De, Shaikh and Biswas [3], Takahashi [9], Tanno [15] and many others.

In a Sasakian manifold the following relations hold: [1]

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{7}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$
$$S(X,\xi) = 2n\eta(X), \tag{9}$$
$$S(\phi X, \phi Y) = S(X,Y) - 2n\eta(X)\eta(Y), \tag{10}$$

for all vector fields X, Y, Z, where S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

A Sasakian manifold is said to be an Einstein manifold if the Ricci tensor S is of the form

$$S(X,Y) = \lambda g(X,Y),$$

where λ is a constant.

Definition 2.1. A Sasakian manifold is said to be a locally $\phi -$ symmetric manifold if [9]

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = 0, \tag{11}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. A Sasakian manifold is said to be a locally pseudo projective $\phi -$ symmetric manifold if

$$\phi^2\left((\nabla_W \tilde{P})(X,Y)Z\right) = 0, \tag{12}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.3. A Sasakian manifold is said to be pseudo projective $\phi -$ recurrent Sasakian manifold if there exists a non-zero $1 -$ form A such that

$$\phi^2((\nabla_W \tilde{P})(X,Y)Z) = A(W)\tilde{P}(X,Y)Z, \tag{13}$$

for arbitrary vector fields X, Y, Z, W, where $\tilde{P}$ is a pseudo projective curvature tensor given by [17]

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{2n+1}\left[\frac{a}{2n} + b\right][g(Y,Z)X - g(X,Z)Y]. \tag{14}$$

where a and b are constants such that a,b≠ 0. If $a = 1$ and $b = -\frac{1}{2n}$ Then (14) takes of the form

$$\tilde{P}(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] = P(X,Y)Z.$$

where P is the projective curvature tensor [21]. Hence the Projective curvature P is a particular case of the tensor $\tilde{P}$. For the reason $\tilde{P}$ is called Pseudo projective curvature tensor, where R is the Riemann curvature tensor S is Ricci tensor and r is the scalar curvature.

If the 1-form A vanishes, then the manifold reduces to a locally pseudo projective $\phi -$ symmetric manifold.

3. Pseudo projective $\phi-$ recurrent Sasakian manifold

In this section we consider a Sasakian manifold which is pseudo projective $\phi-$ recurrent Sasakian manifold. Then by virtue of (1) and (3), we get

$$-(\nabla_W \tilde{P})(X,Y)Z + \eta\left((\nabla_W \tilde{P})(X,Y)Z\right)\xi = A(W)\tilde{P}(X,Y)Z, \quad (15)$$

from which it follows that

$$-g\left((\nabla_W \tilde{P})(X,Y)Z,U\right) + \eta\left((\nabla_W \tilde{P})(X,Y)Z\right)\eta(U) = A(W)g\big(\tilde{P}(X,Y)Z,U\big). \quad (16)$$

Let $\{e_i\}, i = 1,2,\ldots\ldots.2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (16) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(\nabla_W S)(Y,Z) = A(W)\left[S(Y,Z) - \left(\frac{r}{2n+1}\right)g(Y,Z)\right]. \quad (17)$$

Replacing Z by ξ in (17) and using (2) and (9), we get

$$(\nabla_W S)(Y,\xi) = A(W)\left[2n - \left(\frac{r}{2n+1}\right)\right]\eta(Y). \quad (18)$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)$$

using (5), (6) and (9) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = 2ng(\varphi Y,W) + S(Y,\phi W). \quad (19)$$

In view of (18) and (19), we get

$$S(Y,\phi W) = -2ng(\phi Y,W) + A(W)\left[2n - \left(\frac{r}{2n+1}\right)\right]\eta(Y). \quad (20)$$

Replacing Y by ϕY and using (3), (4) and (10) in (20), we get

$$S(Y,W) = 2ng(Y,W). \quad (21)$$

for all Y, W.

Hence, we can state the following theorem:

Theorem 3.1 *A Pseudo projective $\phi-$ recurrent Sasakian manifold (M^{2n+1}, g) is an Einstein manifold.*

Now from (15), we have

$$(\nabla_W \tilde{P})(X,Y)Z = \eta\left((\nabla_W \tilde{P})(X,Y)Z\right)\xi - A(W)\tilde{P}(X,Y)Z. \quad (22)$$

Using (14) in (22), we get

$$\begin{aligned} a(\nabla_W R)(X,Y)Z = {} & a\eta\big((\nabla_W R)(X,Y)Z\big)\xi - aA(W)R(X,Y)Z \\ & + b[(\nabla_W S)(Y,Z)\eta(X) - (\nabla_W S)(X,Z)\eta(Y)]\xi \\ & - b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] \\ & - bA(W)[S(Y,Z)X - S(X,Z)Y] \\ & + \left(\frac{r}{2n+1}\right)\left[\frac{a}{2n} + b\right]A(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \quad (23)$$

From (23) and the Bianchi identity, we get

$$\begin{aligned} & aA(W)\eta(R(X,Y)Z) + aA(X)\eta(R(Y,W)Z) + aA(Y)\eta(R(W,X)Z) \\ & = bA(W)[S(X,Z)\eta(Y) - S(Y,Z)\eta(X)] \\ & \quad - \left(\frac{r}{2n+1}\right)\left[\frac{a}{2n} + b\right]A(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ & \quad + bA(X)[S(Y,Z)\eta(W) - S(W,Z)\eta(Y)] \\ & \quad - \left(\frac{r}{2n+1}\right)\left[\frac{a}{2n} + b\right]A(X)[g(Y,Z)\eta(W) - g(W,Z)\eta(Y)] \\ & \quad + bA(Y)[S(W,Z)\eta(X) - S(X,Z)\eta(W)] \\ & \quad - \left(\frac{r}{2n+1}\right)\left[\frac{a}{2n} + b\right]A(Y)[g(W,Z)\eta(X) - g(X,Z)\eta(W)] \end{aligned} \quad (24)$$

By virtue of (8), we obtain from (24) that

$$\begin{aligned} &aA(W)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \\ &+aA(X)[g(W,Z)\eta(Y)-g(Y,Z)\eta(W)] \\ &+aA(Y)[g(X,Z)\eta(W)-g(W,Z)\eta(X)] \\ &\qquad = bA(W)[S(X,Z)\eta(Y)-S(Y,Z)\eta(X)] \\ &\qquad -\left(\frac{r}{2n+1}\right)\left[\frac{a}{2n}+b\right]A(W)[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)] \\ &\qquad +bA(X)[S(Y,Z)\eta(W)-S(W,Z)\eta(Y)] \\ &\qquad -\left(\frac{r}{2n+1}\right)\left[\frac{a}{2n}+b\right]A(X)[g(Y,Z)\eta(W)-g(W,Z)\eta(Y)] \\ &\qquad +bA(Y)[S(W,Z)\eta(X)-S(X,Z)\eta(W)] \\ &\qquad -\left(\frac{r}{2n+1}\right)\left[\frac{a}{2n}+b\right]A(Y)[g(W,Z)\eta(X)-g(X,Z)\eta(W)] \end{aligned} \tag{25}$$

Putting $Y = Z = e_i$ in (25) and taking summation over i, $1 \le i \le 2n+1$ we get

$$A(W)\eta(X) = A(X)\eta(W), \tag{26}$$

for all vector fields X, W. Replacing X by ξ in (26), we get

$$A(W) = \eta(W)\eta(\rho), \tag{27}$$

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being the vector field associated to the $1-$ form X i.e., $A(X) = g(X,\rho)$. From (27),

we can state the following theorem:

Theorem 3.2 *In a Pseudo projective $\phi-$ Sasakian manifold $(M^{2n+1},g)(n \ge 1)$, the characteristic vector field ξ and the vector field ρ associated to the $1-$ form A are co-directional and the $1-$ form A is given by (27).*

4. On a $3-$ dimensional Locally Pseudo Projective ϕ-Recurrent Sasakian Manifold

On a $3-$ dimensional Sasakian Manifold Ricci tensor and curvature tensor has the following form

$$S(X,Y) = \left(\frac{r}{2}-1\right)g(X,Y) - \left(\frac{r}{2}-3\right)\eta(X)\eta(Y) \tag{28}$$

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r-4}{2}\right)[g(Y,Z)X - g(X,Z)Y] \\ &\quad -\left(\frac{r-6}{2}\right)[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \end{aligned} \tag{29}$$

Taking covariant differentiation of (29), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z \;=\; &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\ &+g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ &-\left(\frac{r-6}{2}\right)[g(Y,Z)(\nabla_W\eta)(X)\xi + g(Y,Z)\eta(X)(\nabla_W\xi) \\ &-g(X,Z)(\nabla_W\eta)(Y)\xi - g(X,Z)\eta(Y)(\nabla_W\xi) \\ &+(\nabla_W\eta)(Y)\eta(Z)X + (\nabla_W\eta)(Z)\eta(Y)X \\ &-(\nabla_W\eta)(X)\eta(Z)Y - (\nabla_W\eta)(Z)\eta(X)Y]. \end{aligned} \tag{30}$$

Taking X, Y, Z, W orthogonal to ξ and using (5) and (6), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z \;=\; &\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ &-\left(\frac{r-6}{2}\right)[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)]\xi \end{aligned} \tag{31}$$

from (31) it follows that

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y] \quad (32)$$

Now, taking X, Y, Z, W orthogonal to ξ and using (1) and (2) in (32), we get

$$\varphi^2(\nabla_W R)(X,Y)Z = -\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \quad (33)$$

Differentiating covariantly (14) with respect to W $(for\ n = 1)$, we get

$$(\nabla_W \tilde{P})(X,Y)Z = a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y] - \frac{dr(W)}{3}\left[\frac{a}{2} + b\right][g(Y,Z)X - g(X,Z)Y]. \quad (34)$$

Using (28)in (34) and then taking X, Y, Z, W orthogonal to ξ, we get

$$(\nabla_W \tilde{P})(X,Y)Z = a(\nabla_W R)(X,Y)Z - \frac{dr(W)}{6}[a-b][g(Y,Z)X - g(X,Z)Y]. \quad (35)$$

Now, applying φ^2 to the both side of (35), we get

$$\phi^2(\nabla_W \tilde{P})(X,Y)Z = a\phi^2(\nabla_W R)(X,Y)Z - [a-b]\frac{dr(W)}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y]. \quad (36)$$

Using (13), (33), (1) in (36), we obtain

$$A(W)\tilde{P}(X,Y)Z = -a\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] + [a-b]\frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi]. \quad (37)$$

taking X, Y, Z orthogonal to W, we get

$$\tilde{P}(X,Y)Z = -\frac{(2a+b)}{6}\frac{dr(W)}{A(W)}[g(Y,Z)X - g(X,Z)Y] \quad (38)$$

Putting $W = \{e_i\}$ in (38), where $\{e_i\}, i = 1,2,3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1 \leq \mathrm{i} \leq 3$, we obtain

$$\tilde{P}(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y].$$

where $\lambda = \left[-\frac{(2a+b)}{6}\frac{dr(e_i)}{A(e_i)}\right]$ is a scalar, since A is a non-zero $1-$ form. Then, by Schur's theorem λ will be a constant on the manifold Hence, we can state the following theorem:

Theorem 4.1 *On a* $3-$ *dimensional locally Pseudo-projective* $\phi-$ *recurrent Sasakian manifold, pseudo-projective curvature tensor is of the form of constant curvature.*

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References

[1] D. E. Blair, "*Contact manifolds in Riemannian geometry*", Lecture Notes in Math, 509 Berlin-Heidelberg-New York, (1976).

[2] T. Q. Binh, L. Tamassy, U.C. De, M. Tarafdar, "*Some remarks on almost Kenmotsu manifolds*", Mathematica Pannonica, 13 (2002), 31-39.

[3] U.C. De, A.A. Shaikh, S. Biswas, "*On* $\phi-$ *Recurrent Sasakian manifolds*", Novi Sad J. Math, 33(2) (2003), 43-48.

[4] U.C. De, G. Pathak, "*On* $3-$ *dimensional Kenmotsu Manifolds*", Indian J. pure appl. Math. 35(2) (2004), 159-165.

[5] U.C. De, A. Yildiz, A.F. Yaliniz, “*On $\phi-$ Recurrent Kenmotsu manifolds*”, Turk J Math, 33 (2009), 17-25.

[6] J.B. Jun , U.C. De, G. Pathak, “*On Kenmotsu manifolds*”, J. Korean Math .Soc., 42 (2005), 435-445.

[7] T. Adati, K. Matsumoto, “*On conformally recurrent and conformally symmetric $P-$ Sasakian manifolds*”, TRU Math., 17 (1977), 25-32.

[8] C. Ozgur, U.C. De, “*On the quasi-conformal curvature tensor of a Kenmotsu manifold*”, Mathematica Pannonica, 17(2) (2006), 221-228.

[9] T. Takahashi, “*Sasakian $\phi-$ symmetric spaces*”, Tohoku Math. J., 29 (1977), 91-113.

[10] K. Yano, “*Concircular geometry*”, Proc. Imp. Acad., Tokyo, 16 (1940), 195-200.

[11] U.C. De, “*On $\phi-$ symmetric Kenmotsu manifolds*”, Int. Electronic J. Geometry, 1(1) (2008), 33-38,

[12] S.S. Shukla, M.K. Shukla, “*On $\phi-$ Symmetric para-Sasakian manifolds*”, Int. Journal of Math. Analysis, 4 (2010), 761-769.

[13] T. Adati, T. Miyazawa, “*On $P-$ Sasakian manifolds satisfying certain conditions*”, Tensor, (N.S.), 33 (1979), 173-178.

[14] I. Sato, “*On a structure similar to the almost contact structure*”, Tensor, (N.S.), 30 (1976), 219-224.

[15] S. Tanno, “*Isometric Immersions of Sasakian manifold in spheres*”, Kodai Math. Sem. Rep., 21, (1969), 448-458.

[16] D. Tarafdar, U.C. De, “*On a type of $P-$ Sasakian manifolds*”, Extracta Mathematicae, 8(1) (1993), 31-36.

[17] B. Prasad, “*A pseudo projective curvature tensor on a Riemannian manifolds*”, Bull. Cal. Math. Soc. 94 (3) (2002), 163-166.

[18] M. Tarafdar, A. Bhattacharyya, D. Debnath, “*A type of pseudo projective $\phi-$ recurrent Trans - Sasakian manifolds*”, Analele stiintifice ale Universitatii AL.I. Cuza Iasi Tomul LII, f-2, S.I, Mathematica (2006), 417-422.

[19] Venkatesha, C.S.Bagewadi, “*On pseudo projective $\phi-$ recurrent Kenmotsu manifolds*”, Soochow Journal of Mathematics, 32(3) (2006), 1-7.

[20] Q. Khan, “*On an Einstein projective Sasakian manifolds*”, NOVI SAD J. Math., 36(1) (2006), 97-102,

[21] R.S. Mishra, “*Structure on a differentiable manifold and their applications*”, Chandrama Prakashan, 50 A, Balrampur House, Allahabad, India (1984).

[22] D. Narain, A. Prakash, B. Prasad, “*A pseudo projective curvature tensor on a lorentzian Para-Sasakian manifolds*”, Analele stiintifice ale Universitatii AL.I. Cuza din iasi (S.N.) Mathematica, Tomul LV, fase.2 (2009), 275-284.