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# A Best Proximity Point Theorem in Metric Spaces with Generalized Distance

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***Abstract***

In this paper at first, we define the weak P-property with respect to a $\tau$-distance such as $p$. Then we state a best proximity point theorem in a complete metric space with generalized distance such that it is an extension of previous research.

**Keywords:** weak P-property, best proximity point, $\tau$-distance, weakly contractive mapping, altering distance functions.

## 1. Introduction

The best proximity point is a interesting topic in best proximity theory. Let $A, B$ be two non-empty subsets of a metric space $(X, d)$ and $T: A \to B$. A solution $x$, for the equation $d(x, T\,x) = d(A, B)$ is called a best proximity point of $T$. If $d(x, T\,x) = 0$ then $x$ is called a fixed point of $T$ [15]. The existence and convergence of best proximity points has generalized by several authors such as Jleli and Samet [3], Prolla [4], Reich [5], Sadiq Basha [7,8], Sehgal and Singh [10,11], Vertivel, Veermani and Bhattacharyya[13] in many directions. On the other hand Suzuki [12] introduced the concept of $\tau$-distance on a metric space. Many fixed point theorems extended for various contractive mappings with respect to a $\tau$-distance. In this paper, by using the concept of $\tau$-distance, we prove a best proximity point theorem. Our results are extension of a best proximity point theorem in metric spaces.

## 2. Preliminary

Let $A, B$ be two non-empty subsets of a metric space $(X, d)$. The following notations will be used throughout this paper:

$d(y, A) \coloneqq \inf\{d(x, y): x \in A\}$,

$d(A, B) \coloneqq \inf\{d(x, y): x \in A, y \in B\}$,

$A_0 \coloneqq \{x \in A: d(x, y) = d(A, B)\ for\ some\ y \in B\}$,

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$B_0 := \{x \in B: d(x,y) = d(A,B)\ for\ some\ x \in A\}$.

We recall that $x \in A$ is a best proximity point of the mapping $T: A \to B$ if $d(x,Tx) = d(A,B)$. It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

**Definition 2.1.**[9] Let $(A,B)$ be a pair of non-empty subsets of a metric space $X$ with $A \neq \emptyset$. Then the pair $(A,B)$ is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{array}\right\} \Rightarrow d(x_1,x_2) = d(y_1,y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset $A$ of $X$, the pair $(A,A)$ has the P-property.

Rhoades [6] introduced a class of contractive mappings called weakly contractive mapping. Harjani and Sadarangani [1] generalized the concept of the weakly contractive mappings in partially ordered metric spaces.

**Definition 2.2.**[2] A function $\psi: [0,\infty) \to [0,\infty)$ is said to be an altering distance function if it satisfies the following conditions:

(i) $\psi$ is continuous and non-decreasing.

(ii) $\psi(t) = 0$ if and only if $t = 0$.

**Definition 2.3.**[6] Let $(X,d)$ be a metric space. $T: X \to X$ is weakly contractive if

$$d(Tx,Ty) \leq d(x,y) - \phi\big(d(x,y)\big), \qquad \forall x,y \in X$$

Where $\phi$ is a altering distance function.

Suzuki [12] introduced the concept of $\tau$-distance on a metric space.

**Definition 2.4.**[12] Let $X$ be a metric space with metric $d$. A function $p: X \times X \to [0,\infty)$ is called $\tau$-distance on X if there exist a function $\eta: X \times [0,\infty) \to [0,\infty)$ such that the following are satisfied:

$(\tau1)$ $p(x,z) \leq p(x,y) + p(y,z) \quad \forall x,y,z \in X$;

$(\tau2)$ $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in [0,\infty)$, and $\eta$ is concave and continuous in it's second variable.

$(\tau3)$ $\lim_n x_n = x$ and $\lim_n \sup\{\eta\big(z_n, p(z_n,x_m)\big): m \geq n\} = 0$ imply $p(w,x) \leq liminf_n p(w,x_n)$ for all $w \in X$;

$(\tau4)$ $\lim_n \sup\{p(x_n,y_m): m \geq n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$;

$(\tau5)$ $\lim_n \eta\big(z_n, p(z_n,x_n)\big) = 0$ and $\lim_n \eta\big(z_n, p(z_n,y_n)\big) = 0$ imply $\lim_n d(x_n,y_n) = 0$.

**Remark 2.5.**[12] It can be replaced $(\tau2)$ by the following $(\tau2)'$.

$(\tau2)'$ $\inf\{\eta(x,t): t > 0\} = 0$ for all $x \in X$ and $\eta$ is non-decreasing in it's second variable.

**Remark 2.6.** If $(X,d)$ is a metric space, then the metric d is a $\tau$-distance on $X$.

In the following examples, we define $\eta: X \times [0,\infty) \to [0,\infty)$ by $\eta(x,t) = t$, for all $x \in X$ and $t \in [0,\infty)$. It is easy to see that $p$ is a $\tau$-distance on metric space $X$.

**Example 2.7.** Let $(X,d)$ be a metric space and $c$ be a positive real number. Then $p: X \times X \to [0,\infty)$ by $p(x,y) = c$ for $x,y \in X$ is a $\tau$-distance on $X$.

**Example 2.8.** Let $(X,\|.\|)$ be a normed space. $p: X \times X \to [0,\infty)$ by $p(x,y) = \|x\| + \|y\|$ for $x,y \in X$ is a $\tau$-distance on $X$.

**Example 2.9.** Let $(X,\|.\|)$ be a normed space. $p: X \times X \to [0,\infty)$ by $p(x,y) = \|y\|$ for $x,y \in X$ is a $\tau$-distance on $X$.

**Definition 2.10.**[12] Let $(X,d)$ be a metric space and $p$ be a $\tau$-distance on $X$. A sequence $\{x_n\}$ in $X$ is a $p$-Cauchy if there exists a function $\eta: X\times[0,\infty)\to[0,\infty)$ satisfying $(\tau2)$-$(\tau5)$ and a sequence $\{z_n\}$ in $X$ such that $\lim_n \sup\{\eta(z_n,p(z_n,x_m)): m\geq n\}=0$.

The following lemmas are essential for the next sections.

**Lemma 2.11.**[12] Let $(X,d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ is a $p$-Cauchy sequence, then it is a Cauchy sequence. Moreover if $\{y_n\}$ is a sequence satisfying $\lim_n \sup\{p(x_n,y_m): m\geq n\}=0$, then $\{y_n\}$ is also $p$-Cauchy sequence and $\lim_n d(x_n,y_n)=0$.

**Lemma 2.12.**[12] Let $(X,d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ in $X$ satisfies $\lim_n p(z,x_n)=0$ for some $z\in X$, then $\{x_n\}$ is a $p$-Cauchy sequence. Moreover if $\{y_n\}$ in $X$ also satisfies $\lim_n p(z,y_n)=0$, then $\lim_n d(x_n,y_n)=0$. In particular, for $x,y,z\in X$, $p(z,x)=0$ and $p(z,y)=0$ imply $x=y$.

**Lemma 2.13.**[12] Let $(X,d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ in $X$ satisfies $\lim_n \sup\{p(x_n,x_m): m\geq n\}=0$, then $\{x_n\}$ is a $p$-Cauchy sequence. Moreover if $\{y_n\}$ in $X$ satisfies $\lim_n p(x_n,y_n)=0$, then $\{y_n\}$ is also $p$-Cauchy sequence and $\lim_n d(x_n,y_n)=0$.

The next result is an immediate consequence of the Lemma 2.11 and Lemma 2.13.

**Corollary 2.14.** Let $(X,d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If a sequence $\{x_n\}$ in $X$ satisfies $\lim_n \sup\{p(x_n,x_m): m\geq n\}=0$, then $\{x_n\}$ is a Cauchy sequence.

## 3. Main results

Inspire of Sankar Raj[9] and Zhang and others[14], we define the weak $P$-property with respect to a $\tau$-distance as follows:

**Definition 3.1.** Let $(A,B)$ be a pair of non-empty subsets of a metric space $(X,d)$ with $A_0\neq\emptyset$. Also let $p$ be a $\tau$-distance on $X$. Then the pair $(A,B)$ is said to have the weak P-property with respect to $p$ if and only if

$$\left.\begin{array}{l} d(x_1,y_1)=d(A,B) \\ d(x_2,y_2)=d(A,B) \end{array}\right\} \Rightarrow p(x_1,x_2)\leq p(y_1,y_2)$$

where $x_1,x_2\in A_0$ and $y_1,y_2\in B_0$.

It is clear that, for any nonempty subset $A$ of $X$, the pair $(A,A)$ has the weak P-property with respect to $p$.

**Remark 3.2.** If $p=d$ then $(A,B)$ is said to have the weak $P$-property where $A\neq\emptyset$. (See [14])

It is easy to see that if $(A,B)$ has the $P$-property then $(A,B)$ has the weak $P$-property.

**Example 3.3.** Let $X=\mathbb{R}^2$ with the usual metric and $p_1,p_2$ be two $\tau$-distances that defined in Example 2.8 and Example 2.9, respectively. Consider,

$A=\{(a,b)\in\mathbb{R}^2 | a=0, 2\leq b\leq 3\}$,

$B=\{(a,b)\in\mathbb{R}^2 | a=1, b\leq 1\}\cup\{(a,b)\in\mathbb{R}^2 | a=1, b\geq 4\}$.

Then $(A,B)$ has the weak $P$-property with respect to $p_1$ and has not the weak $P$-property with respect to $p_2$.

By the definition of $A,B$ we obtain,

$$d((0,2),(1,1))=d((0,3),(1,4))=d(A,B)=\sqrt{2}$$

where $(0,2),(0,3)\in A_0$ and $(1,1),(1,4)\in B_0$. We have,

$p_1((0,2),(0,3))=5$ and $p_1((1,1),(1,4))=\sqrt{2}+\sqrt{17}$,

$p_1((0,3),(0,2))=5$ and $p_1((1,4),(1,1))=\sqrt{17}+\sqrt{2}$.

Therefore $(A,B)$ has the weak P-property with respect to $p_1$. On the other hand, we have

$p_2((0,3),(0,2))=2$ and $p_2((1,4),(1,1))=\sqrt{2}$.

This implies that $(A, B)$ has not the weak $P$-property with respect to $p_2$.

Sankar Raj[9] stated a best proximity point theorem for weakly contractive non-self mappings in metric spaces. The following Theorem is an extension of his results in a metric spaces with generalized distance.

**Theorem 3.4.** Let $A$ and $B$ be non-empty closed subsets of the metric space$(X, d)$ such that $A_0 \neq \emptyset$. Let $p$ be a $\tau$-distance on $X$ and $T: A \to B$ satisfies the following conditions:

(a) $T(A_0) \subseteq B_0$ and $(A, B)$ has the has the weak $P$-property with respect to $p$.

(b) $T$ is a continuous function on $A$ such that

$$\psi\big(p(Tx, Ty)\big) \leq \psi\big(p(x, y)\big) - \phi\big(p(x, y)\big), \ \ \forall x, y \in A$$

where $\psi$ is an altering distance function and $\phi: [0, \infty) \to [0, \infty)$ is non-decreasing function also $\phi(t) = 0$ if and only if $t = 0$.

Then $T$ has a best proximity point in $A$. Moreover, if $d(x, Tx) = d(x^*, Tx^*) = d(A, B)$ for some $x, x^* \in A$, then $p(x, x^*) = 0$.

**Proof.** Choose $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Again, $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B), \ \ \forall n \in \mathbb{N} \cup \{0\}. \tag{1}$$

$(A, B)$ satisfies the weak $P$-property with respect to $p$, therefore from (1) we obtain,

$$p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n), \ \ \forall n \in \mathbb{N}. \tag{2}$$

We will prove that the sequence $\{x_n\}$ is convergent in $A_0$. Since $\psi$ is non-decreasing function we receive that

$$\psi\big(p(x_n, x_{n+1})\big) \leq \psi(p(Tx_{n-1}, Tx_n)), \ \ \forall n \in \mathbb{N}. \tag{3}$$

Also by the definition of $T$, we have

$$\psi(p(Tx_{n-1}, Tx_n) \leq \psi(p(x_{n-1}, x_n)) - \phi(p(x_{n-1}, x_n)), \ \ \forall n \in \mathbb{N}. \tag{4}$$

From (3) and (4), we receive that

$$\begin{aligned}\psi(p(x_n, x_{n+1})) &\leq \psi(p(Tx_{n-1}, Tx_n)) - \phi(p(x_{n-1}, x_n)) \\ &\leq \psi\big(p(x_{n-1}, x_n)\big) - \phi\big(p(x_{n-1}, x_n)\big) \\ &\leq \psi\big(p(x_{n-1}, x_n)\big),\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\psi$ is non-decreasing function, we have

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n), \ \ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\{p(x_n, x_{n+1})\}$ is monotone non-increasing and bounded. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r \geq 0.$$

We claim that $r = 0$. Suppose to the contrary, that $r > 0$. From the inequality

$$\psi(p(x_n, x_{n+1})) \leq \psi\big(p(x_{n-1}, x_n)\big) - \phi\big(p(x_{n-1}, x_n)\big) \leq \psi\big(p(x_{n-1}, x_n)\big),$$

we obtain

$$\lim_{n\to\infty} \phi\big(p(x_{n-1}, x_n)\big) = 0.$$

Since $0 < r \leq p(x_n, x_{n+1})$ and $\phi$ is non-decreasing function,

$$0 < \phi(r) \leq \phi\big(p(x_n, x_{n+1})\big),$$

So,

$$0 < \phi(r) \leq \lim_{n\to\infty} \phi\big(p(x_n, x_{n+1})\big),$$

which is a contradiction. Hence $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Similarly we receive that $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

Now we show that $\lim_{n\to\infty} p(x_n, x_m) = 0$ for $m > n$. In contrary case, there exists $\epsilon > 0$ and two subsequence $\{x_{m_k}\}, \{x_{n_k}\}$ such that $m_k$ is smallest index for which $m_k > n_k > k$, $p\big(x_{n_k}, x_{m_k}\big) \geq \epsilon$. This means that

$$p\left(x_{n_k}, x_{m_k-1}\right) < \epsilon. \tag{5}$$

So, by the triangle inequality and (5), we have

$$\begin{aligned}\epsilon &\le p\left(x_{n_k}, x_{m_k}\right)\\ &\le p\left(x_{n_k}, x_{m_k-1}\right) + p\left(x_{m_k-1}, x_{m_k}\right)\\ &< \epsilon + p\left(x_{m_k-1}, x_{m_k}\right).\end{aligned}$$

Letting $k \to \infty$, we receive that

$$\lim_{k\to\infty} p\left(x_{n_k}, x_{m_k}\right) = \epsilon. \tag{6}$$

By triangle inequality, we have

$$p\left(x_{n_k}, x_{m_k}\right) \le p\left(x_{n_k}, x_{n_k-1}\right) + p\left(x_{n_k-1}, x_{m_k-1}\right) + p\left(x_{m_k-1}, x_{m_k}\right),$$
$$p\left(x_{n_{k}-1}, x_{m_k-1}\right) \le p\left(x_{n_k-1}, x_{n_k}\right) + p\left(x_{n_k}, x_{m_k}\right) + p\left(x_{m_k}, x_{m_k-1}\right).$$

Letting $k \to \infty$ in above two inequality and using (6), we get

$$\lim_{k\to\infty} p\left(x_{n_k-1}, x_{m_k-1}\right) = \epsilon.$$

So,

$$\begin{aligned}0 < \psi(\epsilon) &\le \psi\left(p\left(x_{n_k}, x_{m_k}\right)\right)\\ &\le \psi\left(p\left(Tx_{n_k-1}, Tx_{m_k-1}\right)\right)\\ &\le \psi\left(p\left(x_{n_k-1}, x_{m_k-1}\right)\right) - \phi\left(p\left(x_{n_k-1}, x_{m_k-1}\right)\right)\\ &\le \psi\left(p\left(x_{n_k-1}, x_{m_k-1}\right)\right).\end{aligned}$$

From continuity of $\psi$ in the above inequality, we obtain that

$$\lim_{k\to\infty} \phi\left(p\left(x_{n_k-1}, x_{m_k-1}\right)\right) = 0. \tag{7}$$

From $\lim_{k\to\infty} p\left(x_{n_k-1}, x_{m_k-1}\right) = \epsilon$, we can find $k_0 \in \mathbb{N}$ such that for any $k \ge k_0$,

$$\frac{\epsilon}{2} \le p\left(x_{n_k-1}, x_{m_k-1}\right).$$

This implies that,

$$0 < \phi\left(\frac{\epsilon}{2}\right) \le \phi\left(p\left(x_{n_k-1}, x_{m_k-1}\right)\right), \quad \forall k \ge k_0$$

and this contradicts to (7). Thus $\lim_{n\to\infty} p(x_n, x_m) = 0$ for $m > n$ and this implies that,

$$\lim_n \sup\{p(x_n, x_m) : m \ge n\} = 0.$$

Therefore by Corollary 2.14, $\{x_n\}$ is a Cauchy sequence in $A$. Since $X$ is a complete metric space and $A$ is a closed subset of $X$, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$. $T$ is continuous, therefore with letting $n \to \infty$ in (1), we obtain

$$d(x, Tx) = d(A, B).$$

Now let $x^* \in A$ such that

$$d(x^*, Tx^*) = d(A, B).$$

We claim that $p(x, x^*) = 0$. Suppose to the contrary, that $p(x, x^*) > 0$. Hence $\phi\left(p(x, x^*)\right) > 0$ and therefore by the definition of $T, \psi$, we obtain that,

$$\psi\left(p(x, x^*)\right) \le \psi\left(p(Tx, Tx^*)\right) \le \psi\left(p(x, x^*)\right) - \phi\left(p(x, x^*)\right) \le \psi\left(p(x, x^*)\right),$$

which is a contradiction. Hence $p(x, x^*) = 0$ and this completes the proof of the theorem. ■

The next result is an immediate consequence of the Theorem 3.4 by taking $\psi(t) = 0$ for all $t \ge 0$.

**Corollary 3.5.** Let $A$ and $B$ be non-empty closed subsets of the metric space $(X, d)$ such that $A_0 \neq \emptyset$. Let $p$ be a $\tau$-distance on $X$ and $T: A \to B$ satisfies the following conditions:

(a) $T(A_0) \subseteq B_0$ and $(A, B)$ has the has the weak $P$-property with respect to $p$.

(b) $T$ is a continuous function on $A$ such that

$$p(Tx, Ty) \le p(x, y) - \phi(p(x, y)), \quad \forall x, y \in A$$

where $\phi: [0, \infty) \to [0, \infty)$ is non-decreasing function also $\phi(t) = 0$ if and only if $t = 0$. Then $T$ has a best proximity point in $A$. Moreover, if $d(x, Tx) = d(x^*, Tx^*) = d(A, B)$ for some $x, x^* \in A$, then $p(x, x^*) = 0$.

The following result is the special case of the Corollary 3.5, obtained by setting $p = d$.

**Corollary 3.6.**[9] Let $(A, B)$ be a pair of two nonempty, closed subsets of a complete metric space $X$ such that $A_0$ is non-empty. Let $T: A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$ . Assume that the pair $(A, B)$ has the $P$-property. Then there exists a unique $x^*$ in A such that $d(x^*, Tx^*) = d(A, B)$.

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