Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 17 (2017), 179–193
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

The Chebyshev collocation solution of the time fractional coupled Burgers' equation

Basim Albuohimad, Hojatollah Adibi*

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., 15914, Tehran, Iran.

Abstract

This paper is concerned with the numerical solution of the time fractional coupled Burgers' equation. The proposed hybrid solution is based on Chebyshev collection method for space variable, and the trapezoidal quadrature technique. Finally the error analysis is discussed and some test examples are presented to demonstrate the applicability and efficiency of the method.

Keywords: Fractional coupled Burgers' equation, trapezoidal quadrature, finite difference, Chebyshev polynomials, spectral collection method.
2010 MSC: 41A10, 65D30, 35K40.

1. Introduction

The fractional calculus [8, 13, 21, 23, 25, 26] is an important branch of applied mathematics. This type of differentiation and integration could be considered as generalization to the usual definition of differentiation and integration to non-integer order [1, 3, 4]. Fractional partial differential equations have recently been applied to different areas of sciences, mathematics, physics, chemistry, engineering, continuum, statistical mechanics, and dynamic system [2, 5, 8, 14, 23, 29, 33, 34].

In this paper, we study the coupled Burgers' equation with time-fractional derivatives given as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial (uv)}{\partial x} + f(x,t),$$
$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial (uv)}{\partial x} + g(x,t).$$

The Burgers' equation can be linearized by Hopf-Cole transformation [12]. Mathematical models of essential flow equations describing unsteady transport issues consist of systems of nonlinear parabolic and hyperbolic partial differential equations. The coupled Burgers' equations form an important type of

*Corresponding author
Email addresses: basim.albuohimad@gmail.com (Basim Albuohimad), adibih@aut.ac.ir (Hojatollah Adibi)

doi:10.22436/jmcs.017.01.16

Received 2016-10-30

such partial differential equations. These equations happen in a large number of physical problems such as the phenomena of turbulence flow through a shock wave traveling in a viscous fluid (see [6, 24]).

In recent years, many researchers have studied the fractional partial differential equations and dealt with the fractional Burgers' equation utilizing different techniques [9, 16, 17, 19, 20, 27, 28, 32, 35]. More recently, the authors in [15] applied the Chebyshev polynomials expansion method to find both analytical and numerical solutions of the fractional transport equation in the one dimensional geometry. Dehghan et al. [10] studied Burgers' equation by using novel semi-analytical methods such as the homotopy perturbation method. Also, the solution of two dimensional Burgers' equation based on operational matrices was presented in [18].

In the present paper, we use the spectral collection method based on orthogonal Chebyshev polynomials, and trapezoidal quadrature (TQ) and finite difference method (FDM) by Caputo derivative to solve the system of coupled Burgers' equation.

2. Definitions and basic properties

In this section, we give some basic notions about fractional calculus and Chebyshev polynomials, which are required for our subsequent development.

2.1. Fractional derivatives

Here we recall definitions and basic results about the fractional calculus. For more details we refer to [26].

Definition 2.1. A real function $u(t),\ t > 0$ is said to be in space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $u(t) = t^p u_1(t)$, where $u_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $u^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu,\ \mu \geqslant -1$, is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad \alpha > 0,$$
$$I^0 u(t) = u(t),$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 2.3. The fractional derivative of $u(t)$ in the Caputo sense is defined as

$$D^\alpha u(t) = I^{m-\alpha} D^m u(t),$$

for $m-1 < \alpha \leqslant m$, $m \in \mathbb{N}$, $t > 0$ and $u \in C_{-1}^m$. Also it can be rewritten in the following form

$$D^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} u^{(m)}(x) dx.$$

Similar to integer-order differentiation, Caputo fractional differential has the linear property:

$$D^\alpha \left(c_1 f_1(t) + c_2 f_2(t)\right) = c_1 D^\alpha f_1(t) + c_2 D^\alpha f_2(t),$$

where c_1 and c_2 are constants. If so, for Caputo derivative we have the following basic properties,

i)

$$D^\alpha t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma \geqslant \lceil \alpha \rceil \text{ or } \gamma \notin \mathbb{N} \text{ and } \gamma > \lfloor \alpha \rfloor, \\ 0, & \text{for } \gamma \in \mathbb{N}_0, \end{cases}$$

ii) $D^{\alpha}(c) = 0$,

iii)

$$I^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{i=0}^{m-1} \frac{u^{(i)}(0)}{i!} t^{i}, \tag{2.1}$$

where c is constant, $\lfloor \alpha \rfloor$ and $\lceil \alpha \rceil$ are floor and ceiling functions, respectively, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $\mathbb{N} = \{1, 2, \ldots\}$.

2.2. *Chebyshev polynomials*

The well-known Chebyshev polynomial of the first kind of degree n, which are defined on interval $[-1, 1]$ are given by [5]:

$$T_n(x) = \cos(n \cos^{-1}(x)), \qquad n = 0, 1, 2, \cdots,$$

by setting $x = \cos(\theta)$, we have:

$$T_n(x) = \cos(n\theta),$$

hence, we have the relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, \cdots.$$

Definition 2.4 (Shifted Chebyshev polynomials)**.** Shifted Chebyshev polynomials of the first kind denoted by $T_n^*(x)$ are defined as [30]

$$T_n^*(x) = T_n(\frac{2x}{L} - 1), \quad 0 \leqslant x \leqslant L.$$

Shifted Chebyshev polynomials $T_n^*(x)$ satisfy in the recurrence relation:

$$T_n^*(x) = 2(\tfrac{2x}{L} - 1)T_{n-1}^*(x) - T_{n-2}^*(x),$$

with the starting value, $T_0^*(x) = 1$, $T_1^*(x) = \frac{2x}{L} - 1$.

These polynomials are orthogonal with respect to the weight function $w(x) = (Lx - x^2)^{-1/2}$, $0 \leqslant x \leqslant L$.

Also, shifted Chebyshev polynomials may be represented by:

$$T_n^*(x) = n \sum_{j=0}^{n} (-1)^{n-j} \frac{(n+j-1)!}{L^j (2j)!(n-j)!} 2^{2j} x^j. \tag{2.2}$$

2.3. *Shifted Chebyshev polynomials derivatives*

The derivative formula for the shifted Chebyshev polynomial, via Eq. (2.2), is given by

$$\begin{aligned} D^m T_n^*(x) &= n \sum_{j=0}^{n} (-1)^{n-j} \frac{(n+j-1)!}{L^j (2j)!(n-j)!} 2^{2j} D^m (x)^j \\ &= n \sum_{j=m}^{n} (-1)^{n-j} \frac{(n+j-1)!(j!)}{L^j (2j)!(n-j)!(j-m)!} 2^{2j} (x)^{j-m}, \end{aligned} \tag{2.3}$$

where m is positive integer number.

An arbitrary function $u(x)$ can be approximated in the interval $[0, L]$ with shifted Chebyshev polynomials by the formula $u_n(x) = \sum_{i=0}^{n} a_i T_i^*(x)$ and then we can write

$$D^m(u_n(x)) = D^m \left[\sum_{i=0}^{n} a_i T_i^*(x) \right] = \sum_{i=m}^{n} a_i D^m(T_i^*(x)) .$$

Using Eq. (2.3), we have:

$$D^m(u_n(x)) = \sum_{i=m}^{n} i a_i \sum_{j=k}^{i} (-1)^{i-j} \frac{(i+j-1)!(j!)}{L^i(2j)!(i-j)!(j-m)!} 2^{2j} x^{j-m} = \sum_{i=m}^{n} \sum_{j=k}^{i} a_i w_{i,j}^{(m)} x^{j-m},$$

where

$$w_{i,j}^{(m)} = (-1)^{i-j} \frac{i(i+j-1)!(j!)}{L^i(2j)!(i-j)!(j-m)!} 2^{2j}.$$

For solving the time-fractional partial differential equations, we apply the Chebyshev polynomials collection method on space variable, which gives the exponential convergence rate on space.

3. Function approximation

Let $w(x) = (1-x^2)^{-1/2}$ which denotes a non-negative, integrable, real-valued function over the interval $\Lambda = (-1, 1)$. We define

$$L_w^2(\Lambda) = \left\{ v : \Lambda \longrightarrow \mathbb{R} \;\middle|\; v \text{ is measurable and } \|v\|_w < \infty \right\},$$

where

$$\|v\|_w^2 = \int_{-1}^{1} v^2(x)\, w(x)\, dx$$

is the norm induced by the inner product of the space $L_w^2(\Lambda)$,

$$\langle u, v \rangle_w = \int_{-1}^{1} u(x)\, v(x)\, w(x)\, dx. \tag{3.1}$$

It is easily seen that $\left\{ T_j(x) \right\}_{j \geqslant 0}$ denotes a system which is mutually orthogonal under (3.1), i.e.,

$$\langle T_n(x), T_m(x) \rangle_w = c_n \delta_{nm}, \qquad c_0 = \pi, \quad c_n = \frac{\pi}{2}, \quad n \geqslant 1.$$

The classical Weierstrass theorem [31] implies that such a system is complete in the space $L_w^2(\Lambda)$. Thus, for any function $u(x) \in L_w^2(\Lambda)$ the following expansion holds

$$u(x) = \sum_{j=0}^{+\infty} a_j T_j(x), \tag{3.2}$$

where

$$a_j = c_j^{-1} \int_{-1}^{1} u(x) T_j(x) w(x) dx = c_j^{-1} \langle u(x), T_i(x) \rangle_w. \tag{3.3}$$

If $u(x)$ in the Eq. (3.2) is truncated up to the m-th terms, then it can be written as

$$u(x) \simeq u_m(x) = \sum_{j=0}^{m} a_j T_j(x). \tag{3.4}$$

Now, we can estimate an upper bound for function approximation in a special case. Firstly, the error function can be defined in the following form

$$e_m(x) = u(x) - u_m(x), \qquad x \in \Lambda.$$

Accordingly, the maximum error bound for $T_n(x)$ will be as:

$$E_m^\infty = \|e_m(x)\|_\infty = \max_{x\in\Lambda} \Big|\sum_{j=m+1}^{\infty} a_j T_j(x)\Big| = \max_{0\leqslant\theta\leqslant 2\pi} \Big|\sum_{j=m+1}^{\infty} a_j \cos(j\theta)\Big| \leqslant \sum_{j=m+1}^{\infty} |a_j|. \tag{3.5}$$

If so, the completeness of the system $\Big\{T_i(x)\Big\}_{i\geqslant 0}$ is achieved by virtue of:

$$u_m(x) \longrightarrow u(x), \quad \|e_m(x)\|_w \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty\,.$$

Lemma 3.1. *The Chebyshev norm error can be rewritten as*

$$(E_m^w)^2 = \|e_m(x)\|_w^2 = \frac{2}{\pi}\sum_{i=m+1}^{\infty} \langle u(x), T_i(x)\rangle_w^2.$$

Proof. The completeness of the system $\Big\{T_i(x)\Big\}_{i\geqslant 0}$ helped us to consider the error as

$$(E_m^w)^2 = \|\sum_{i=m+1}^{\infty} a_i T_i(x)\|_w^2.$$

Using the definition of $\|.\|_w$, one has

$$(E_m^w)^2 = \sum_{i=m+1}^{\infty}\sum_{j=m+1}^{\infty} a_i a_j \langle T_i(x), T_j(x)\rangle_w = \frac{\pi}{2}\sum_{i=m+1}^{\infty}\sum_{j=m+1}^{\infty} a_i a_j \delta_{ij} = \frac{\pi}{2}\sum_{j=m+1}^{\infty} a_j^2,$$

and consequently, Eq. (3.3) proves the lemma. □

This lemma shows that the convergence rate is involved with the function $u(x)$. Now, by knowing that the function $u(x) \in L_w^2(\Lambda)$ has some good properties, we could present an upper bound for estimating the error of function approximation by this basis function.

Theorem 3.2. *Let $u_m(x)$ be a function approximation of $u(x) \in L_w^2(\Lambda)$ obtained by (3.4) and be analytic on Λ, then an error bound for this approximation can be presented as follows:*

$$E_m^\infty \leqslant M_\infty \frac{1}{(m+1)!}\left(\frac{1}{2}\right)^m, \qquad E_m^w \leqslant \sqrt{\frac{\pi}{3}} M_\infty \frac{1}{(m+1)!}\left(\frac{1}{2}\right)^{m+\frac{1}{2}},$$

where $M_\infty \geqslant 2\max_i |u^{(i)}(x)|$, $x \in (-1,1)$.

Proof. Using Eq. (3.3), and knowing that $u(x)$ is analytic, we have

$$\frac{\pi}{2} a_i = \sum_{j=0}^{i-1} \frac{u^{(j)}(0)}{j!}\int_{-1}^{1} x^j T_i(x) w(x)\,dx + \frac{u^{(i)}(\eta_i)}{i!}\int_{-1}^{1} x^i T_i(x) w(x)\,dx, \qquad \eta_i \in (-1,1).$$

Also, using the following property of Chebyshev polynomials

$$\int_{-1}^{1} x^j T_i(x)w(x)dx = 0, \quad j < i, \qquad\qquad \int_{-1}^{1} x^i T_i(x)w(x)dx = \frac{\pi}{2^i},$$

we can write $a_i = \frac{2u^{(i)}(\eta_i)}{i!2^i}$.

Now assuming $M_\infty \geqslant 2\max_i |u^{(i)}(x)|$, $x \in (-1, 1)$ and using Eq. (3.5) give

$$E_m^\infty \leqslant M_\infty \sum_{i=m+1}^{\infty} \frac{1}{i!2^i} \leqslant M_\infty \frac{1}{(m+1)!2^m}.$$

Also, according to Lemma 3.1, we can prove the theorem, as

$$(E_m^w)^2 \leqslant \frac{\pi}{2} M_\infty^2 \sum_{i=m+1}^{\infty} \frac{1}{(i!)^2 2^{2i}} \leqslant \pi M_\infty^2 \frac{1}{((m+1)!)^2} \sum_{i=m+1}^{\infty} \frac{1}{2^{2i+1}},$$

$$E_m^w \leqslant \sqrt{\frac{\pi}{3}} M_\infty \frac{1}{(m+1)!} \left(\frac{1}{2}\right)^{m+\frac{1}{2}}.$$

□

If $u(x)$ is finite times continuously differentiable, some bounds for truncation error have been presented by [7].

The natural Sobolev norms in which to measure approximation errors for the Chebyshev system involves the Chebyshev weight in the quadratic averages of the error and its derivatives over the interval Λ. Thus, we set

$$\|u\|_{H_w^r(\Lambda)} = \left(\sum_{k=0}^{r} \|u^{(k)}\|_w^2 \right)^{\frac{1}{2}}.$$

The Hilbert space associated to this norm is denoted by $H_w^r(\Lambda)$. We also define the seminorms

$$|u|_{H_w^{r;m}(\Lambda)} = \left(\sum_{k=\min(r,m+1)}^{r} \|u^{(k)}\|_w^2 \right)^{\frac{1}{2}}.$$

Theorem 3.3. *For all* $u \in H_w^r(\Lambda)$*, with* $r \geqslant l \geqslant 1$*, the truncation error* $e_m(x)$ *satisfies the inequalities*

$$E_m^w \leqslant Cm^{-r} |u|_{H_w^{r;m}(\Lambda)} \quad \text{and} \quad \|e_m\|_{H_w^l(\Lambda)} \leqslant Cm^{l-r} |u|_{H_w^{r;m}(\Lambda)}.$$

Proof. See [7]. □

4. Trapezoidal quadrature formula

Now we recognize the following fractional differential equation,

$$D_*^\alpha u(t) = f(u(t), t), \quad u(0) = u_0, \quad 0 < \alpha < 1,$$

which by applying Eq. (2.1) converts to the Volterra integral equation,

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(u(s), s) ds. \tag{4.1}$$

For the numerical computation of (4.1), the integral is replaced by the trapezoidal quadrature formula at point t_n

$$\int_0^{t_n} (t_n - s)^{\alpha-1} g(s) ds \approx \int_0^{t_n} (t_n - s)^{\alpha-1} \tilde{g}_n(s) ds, \tag{4.2}$$

where $g(s) = f(s, u(s))$ and $\tilde{g}_n(s)$ is the piecewise linear interpolation of g with nodes and knots chosen at $t_j, j = 0, 1, 2, ..., n$. After some elementary calculations, the right hand side of (4.2) gives [11]

$$\int_0^{t_n} (t_n - s)^{\alpha-1} \tilde{g}_n(s) ds = \frac{\tau^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} k_{j,n}^{(\alpha)} g(t_j), \tag{4.3}$$

where

$$k_{j,n}^{(\alpha)} = \begin{cases} (n-1)^{\alpha+1} - (n-1-\alpha)n^\alpha, & \text{if } j = 0, \\ (n-j+1)^{\alpha+1} + (n-j-1)^{\alpha+1} - 2(n-j)^{\alpha+1}, & \text{if } 1 \leqslant j \leqslant n-1, \\ 1, & \text{if } j = n, \end{cases}$$

and $k_{j,n}^{(\alpha)}$ are positive number and bounded $(0 < k_{j,n}^{(\alpha)} \leqslant 1)$.

From (4.2) we immediately get

$$\left| \int_0^{t_n} (t_n - s)^{\alpha-1} g(s) ds - \int_0^{t_n} (t_n - s)^{\alpha-1} \tilde{g}_n(s) ds \right| \leqslant \max_{0 \leqslant t \leqslant t_n} \left| g(t) - \tilde{g}_n(t) \right| \int_0^{t_n} |(t_n - s)^{\alpha-1}| ds,$$

so that error bounds and orders of convergence for product integration follow from standard results of approximation theory. For a piecewise linear approximation to a smooth function $g(t)$, the product trapezoidal is of second order [22].

The time-fractional coupled Burgers' equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = L_1[u(x,t), v(x,t)],$$
$$\frac{\partial^\beta v(x,t)}{\partial t^\beta} = L_2[u(x,t), v(x,t)],$$

with the initial condition $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ can be converted to the following singular integro-partial differential equation

$$u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_1[u(x,s), v(x,s)] ds,$$

$$v(x,t) = v_0(x) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} L_2[u(x,s), v(x,s)] ds.$$

Then, applying the trapezoidal quadrature formula (4.3), yields

$$u(x,t_n) = u_0(x) + \frac{\tau^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} k_{j,n}^{(\alpha)} L_1[u(x,t_j), v(x,t_j)] ds, \tag{4.4}$$

$$v(x,t_n) = v_0(x) + \frac{\tau^\beta}{\Gamma(\beta+2)} \sum_{j=0}^{n} k_{j,n}^{(\beta)} L_2[u(x,t_j), v(x,t_j)] ds. \tag{4.5}$$

The above space differential equations are independent of time variable and can be solved iteratively according to sufficient boundary conditions. The rate of convergence of this formula is $O(\tau^2)$ on time variable.

5. Finite difference approximations for time-fractional derivative

In this section, a fractional order finite difference approximation [21] for the time fractional partial differential equations is proposed.

Define $t_k = k\tau, k = 0, 1, 2, ..., n$, where $\tau = T/n$. The time fractional derivative term of order $0 < \alpha \leqslant 1$ with respect to time at $t = t_n$ is approximated by the following scheme,

$$\begin{aligned}\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t_n}(t_n-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds\\ &\approx \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n-s)^{-\alpha}\frac{u(x,t_{k+1})-u(x,t_k)}{\tau}\\ &= \sum_{k=0}^{n-1} w_{n-k-1}^{(\alpha)}(u(x,t_{k+1})-u(x,t_k)).\end{aligned} \tag{5.1}$$

Similarly,

$$\frac{\partial^\beta v(x,t_n)}{\partial t^\beta} = \sum_{k=0}^{n-1} w_{n-k-1}^{(\beta)}(v(x,t_{k+1})-v(x,t_k)),$$

where,

$$\begin{aligned}w_k^{(\alpha)} &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}((k+1)^{1-\alpha}-k^{1-\alpha}),\\ w_k^{(\beta)} &= \frac{\tau^{-\beta}}{\Gamma(2-\beta)}((k+1)^{1-\beta}-k^{1-\beta}).\end{aligned}$$

We apply this formula to discretize the time variable. The rate of convergence of this formula is $O(\tau^{2-\alpha})$.

6. Collection method to solve time-fractional coupled Burgers' equation

In this paper, we decide to use the spectral collocation methods and in addition, trapezoidal formula or finite difference formula to solve time-fractional coupled Burgers' equation of the form:

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial (uv)}{\partial x} + f(x,t),\\ \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} &= \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial (uv)}{\partial x} + g(x,t),\end{aligned}$$

with initial conditions

$$u(x,0) = f_1(x), \quad v(x,0) = g_1(x),$$

and the boundary conditions

$$\begin{aligned}u(0,t) = a_1(t),\ u(L,t) = b_1(t),\\ v(0,t) = a_2(t),\ v(L,t) = b_2(t),\end{aligned}$$

where D_t^α , D_t^β denote the Caputo fractional derivatives of orders α and β with respect to t, respectively and $u(x,t)$ and $v(x,t)$ are unknown functions. For $t = t_n$ the functions $u(x,t_n)$ and $v(x,t_n)$ are discretized in time and they can be expanded as

$$u(x,t_n) \simeq \sum_{i=0}^{m} a_i^n T_i^*(x), \qquad v(x,t_n) \simeq \sum_{i=0}^{m} b_i^n T_i^*(x), \tag{6.1}$$

where the functions $T_i^*(x),\ i = 0, 1, ..., m$ can be chosen as shifted Chebyshev polynomials in $[0, L]$. The time fractional derivative can be discretized by trapezoidal formula (4.4) and finite difference approximation (5.1).

7. Numerical experiments

In this section we present four examples to illustrate the numerical results.

Example 7.1. We consider the following time fractional coupled Burgers' equation,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial (uv)}{\partial x} + f(x,t),$$
$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial (uv)}{\partial x} + g(x,t),$$

by initial conditions

$$u(x,0) = 0, \quad v(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = 0, \;\; u(1,t) = t^3,$$
$$v(0,t) = 0, \;\; v(1,t) = t^3,$$

where $f(x,t)$ and $g(x,t)$ are given by

$$f(x,t) = \frac{6xt^{3-\alpha}}{\Gamma(4-\alpha)}, \qquad g(x,t) = \frac{6xt^{3-\beta}}{\Gamma(4-\beta)}.$$

Exact solution of this problem is $u(x,t) = v(x,t) = xt^3$.

First, we approximate $u(x,t)$ and $v(x,t)$ by forms of Eq. (6.1), hence, we have

$$\frac{\partial u_m(x,t_n)}{\partial x} = \sum_{i=0}^{m} a_i^n T_i^{*\prime}(x), \qquad \frac{\partial v_m(x,t_n)}{\partial x} = \sum_{i=0}^{m} b_i^n T_i^{*\prime}(x),$$
$$\frac{\partial^2 u_m(x,t_n)}{\partial x^2} = \sum_{i=0}^{m} a_i^n T_i^{*\prime\prime}(x), \qquad \frac{\partial^2 v_m(x,t_n)}{\partial x^2} = \sum_{i=0}^{m} b_i^n T_i^{*\prime\prime}(x).$$

Now, we can solve this problem by using spectral collection method with the basic principles of shifted Chebyshev polynomials $T_i^*(x)$ at roots x_r with two methods for time-fractional derivative for $D^\alpha u, D^\beta v$.

Trapezoidal formula. We apply trapezoidal formula in Eqs. (4.4) and (4.5) for this problem, and in addition to that we use the initial and boundary conditions, and by using approximates $u(x,t)$ and $v(x,t)$, we have

$$\sum_{i=0}^{m} a_i^n T_i^*(x_r) = u_0(x_r) + \frac{\tau^\alpha}{\Gamma(2+\alpha)} \sum_{j=0}^{n} k_{j,n}^{(\alpha)} L_1[u_m(x,t_j), v_m(x,t_j)] \tag{7.1}$$

$$\sum_{i=0}^{m} b_i^n T_i^*(x_r) = v_0(x_r) + \frac{\tau^\beta}{\Gamma(2+\beta)} \sum_{j=0}^{n} k_{j,n}^{(\beta)} L_2[u_m(x,t_j), v_m(x,t_j)], \tag{7.2}$$

where,

$$\begin{aligned} L_1[u_m(x,t_j),v_m(x,t_j)] = &\sum_{i=0}^{m} a_i^j T_i^{*''}(x_r) + 2\sum_{i=0}^{m} a_i^j T_i^{*}(x_r)\sum_{i=0}^{m} a_i^j T_i^{*'}(x_r) - \sum_{i=0}^{m} a_i^j T_i^{*}(x_r)\sum_{i=0}^{m} b_i^j T_i^{*'}(x_r) \\ &- \sum_{i=0}^{m} b_i^j T_i^{*}(x_r)\sum_{i=0}^{m} a_i^j T_i^{*'}(x_r) + f(x_r,t_j), \\ L_2[u_m(x,t_j),v_m(x,t_j)] = &\sum_{i=0}^{m} b_i^j T_i^{*''}(x_r) + 2\sum_{i=0}^{m} b_i^j T_i^{*}(x_r)\sum_{i=0}^{m} b_i^j T_i^{*'}(x_r) - \sum_{i=0}^{m} a_i^j T_i^{*}(x_r)\sum_{i=0}^{m} b_i^j T_i^{*'}(x_r) \\ &- \sum_{i=0}^{m} b_i^j T_i^{*}(x_r)\sum_{i=0}^{m} a_i^j T_i^{*'}(x_r) + g(x_r,t_j), \end{aligned}$$

and

$$\sum_{i=0}^{m}(-1)^i a_i^n = 0, \qquad \sum_{i=0}^{m} a_i^n = t_n^3,$$
$$\sum_{i=0}^{m}(-1)^i b_i^n = 0, \qquad \sum_{i=0}^{m} b_i^n = t_n^3,$$

where $x_1, ..., x_{m-1}$ are the roots of $T^*_{m-1}(x)$, then by substituting this roots in Eqs. (7.1) and (7.2), hence we have system of equations which is numerical solution of the time fractional coupled Burgers' equation.

Finite difference method. To do so, we use (5.1) along with the initial and boundary conditions and deduce

$$\begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{k=0}^{n-1} w_{n-k-1}^{(\alpha)}\left(\sum_{i=0}^{m}\left(a_i^{k+1}-a_i^k\right)T_i^*(x_r)\right)\right] - \sum_{i=0}^{m} a_i^n T_i^{*''}(x_r) - 2\sum_{i=0}^{m} a_i^n T_i^{*}(x_r)\sum_{i=0}^{m} a_i^n T_i^{*'}(x_r) \\ &\quad + \sum_{i=0}^{m} a_i^n T_i^{*}(x_r)\sum_{i=0}^{m} b_i^n T_i^{*'}(x_r) + \sum_{i=0}^{m} b_i^n T_i^{*}(x_r)\sum_{i=0}^{m} a_i^n T_i^{*'}(x_r) - f(x_r,t_n) = 0, \end{aligned} \tag{7.3}$$

$$\begin{aligned} &\frac{\tau^{-\beta}}{\Gamma(2-\beta)}\left[\sum_{k=0}^{n-1} w_{n-k-1}^{(\beta)}\left(\sum_{i=0}^{m}\left(b_i^{k+1}-b_i^k\right)T_i^*(x_r)\right)\right] - \sum_{i=0}^{m} b_i^n T_i^{*''}(x_r) - 2\sum_{i=0}^{m} b_i^n T_i^{*}(x_r)\sum_{i=0}^{m} b_i^n T_i^{*'}(x_r) \\ &\quad + \sum_{i=0}^{m} a_i^n T_i^{*}(x_r)\sum_{i=0}^{m} b_i^n T_i^{*'}(x_r) + \sum_{i=0}^{m} b_i^n T_i^{*}(x_r)\sum_{i=0}^{m} a_i^n T_i^{*'}(x_r) - g(x_r,t_n) = 0, \end{aligned} \tag{7.4}$$

and

$$\sum_{i=0}^{m}(-1)^i a_i^n = 0, \qquad \sum_{i=0}^{m} a_i^n = t_n^3,$$
$$\sum_{i=0}^{m}(-1)^i b_i^n = 0, \qquad \sum_{i=0}^{m} b_i^n = t_n^3,$$

where $x_1, ..., x_{m-1}$ are the roots of $T^*_{m-1}(x)$.

Then, by substituting the roots, above, in the Eqs. (7.3) and (7.4), we will have a system of equations.

The maximum absolute errors E_5^{∞} are reported and compared between hybrid collection method and finite difference method in Tables 1 and 2 and Figure 1.

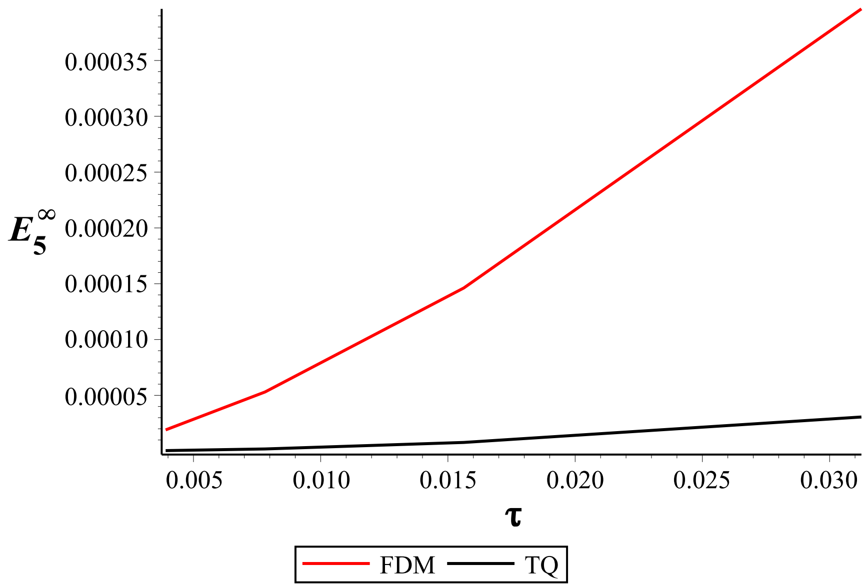

Figure 1: Graph of the comparison for the Maximum absolute errors between TQ and FDM.

Table 1: Compared errors between methods for Example 7.1 with $m = 5,\ \alpha = \beta = 0.5,\ L = 1$.

τ	Collection with TQ		Collection with FDM	
	$E_5^\infty(u)$	$E_5^\infty(v)$	$E_5^\infty(u)$	$E_5^\infty(v)$
0.03125	$3.05993758 \times 10^{-5}$	$3.05993758 \times 10^{-5}$	$3.96243489 \times 10^{-4}$	$3.96243489 \times 10^{-4}$
0.015625	$7.80684685 \times 10^{-6}$	$7.80684685 \times 10^{-6}$	$1.46199451 \times 10^{-4}$	$1.46199451 \times 10^{-4}$
0.0078125	$1.97584942 \times 10^{-6}$	$1.97584942 \times 10^{-6}$	$5.30198057 \times 10^{-5}$	$5.30198057 \times 10^{-5}$
0.00390625	$4.97781868 \times 10^{-7}$	$4.97781868 \times 10^{-7}$	$1.90424033 \times 10^{-5}$	$1.90424033 \times 10^{-5}$
0.001953125	$1.25065007 \times 10^{-7}$	$1.25065007 \times 10^{-7}$	$6.80038150 \times 10^{-6}$	$6.80038150 \times 10^{-6}$

Table 2: Compared errors between methods for Example 7.1 with $\tau = 1/1000,\ \alpha = \beta = 0.5,\ L = 1$.

m	Collection with TQ		Collection with FDM	
	$E_m^\infty(u)$	$E_m^\infty(v)$	$E_m^\infty(u)$	$E_m^\infty(v)$
3	$3.20149740 \times 10^{-8}$	$3.20149740 \times 10^{-8}$	$2.41902729 \times 10^{-6}$	$2.41902729 \times 10^{-6}$
4	$3.26287348 \times 10^{-8}$	$3.26287348 \times 10^{-8}$	$2.50951941 \times 10^{-6}$	$2.50951941 \times 10^{-6}$
5	$3.28895893 \times 10^{-8}$	$3.28895893 \times 10^{-8}$	$2.60731015 \times 10^{-6}$	$2.60731015 \times 10^{-6}$

Example 7.2. We consider the time fractional coupled Burgers' equation with initial conditions

$$u(x,0) = 0, \qquad v(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0.8414t^3,$$
$$v(0,t) = 0, \quad v(1,t) = 0.8414t^3,$$

where $f(x,t)$ and $g(x,t)$ are given by

$$f(x,t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} \sin(x) + t^3 \sin(x),$$
$$g(x,t) = \frac{\Gamma(4)}{\Gamma(4-\beta)} t^{3-\beta} \sin(x) + t^3 \sin(x).$$

Exact solution for this problem is $u(x,t) = v(x,t) = t^3 \sin(x)$.

The maximum absolute errors are reported and compared between hybrid collection method and finite difference method in Tables 3 and 4.

Table 3: Compared errors between methods for Example 7.2 with $m = 5$, $\alpha = \beta = 0.5$, $L = 1$.

τ	Collection with TQ		Collection with FDM	
	$E_5^\infty(u)$	$E_5^\infty(v)$	$E_5^\infty(u)$	$E_5^\infty(v)$
0.125	$4.11929356 \times 10^{-4}$	$4.11929356 \times 10^{-4}$	$2.38860019 \times 10^{-3}$	$2.38860019 \times 10^{-3}$
0.0625	$1.08023735 \times 10^{-4}$	$1.08023735 \times 10^{-4}$	$9.71149782 \times 10^{-4}$	$9.71149782 \times 10^{-4}$
0.03125	$2.60198059 \times 10^{-5}$	$2.60198059 \times 10^{-5}$	$3.68124891 \times 10^{-4}$	$3.68124891 \times 10^{-4}$
0.015625	$6.05915237 \times 10^{-6}$	$6.05915237 \times 10^{-6}$	$1.33717524 \times 10^{-4}$	$1.33717524 \times 10^{-4}$

Table 4: Compared errors between methods for Example 7.2 with $\tau = 1/128$, $\alpha = \beta = 0.5$, $L = 1$.

m	Collection with TQ		Collection with FDM	
	$E_m^\infty(u)$	$E_m^\infty(v)$	$E_m^\infty(u)$	$E_m^\infty(v)$
3	$2.22390450 \times 10^{-3}$	$2.22390450 \times 10^{-3}$	$2.16075055 \times 10^{-3}$	$2.16075055 \times 10^{-3}$
4	$1.06778378 \times 10^{-4}$	$1.06778378 \times 10^{-4}$	$1.41457658 \times 10^{-4}$	$1.41457658 \times 10^{-4}$
5	$2.00011705 \times 10^{-6}$	$2.00011705 \times 10^{-6}$	$4.69272546 \times 10^{-5}$	$4.69272546 \times 10^{-5}$

Example 7.3. We consider the time fractional coupled Burgers' equation of Example 7.1 with initial conditions

$$u(x,0) = 0, \qquad v(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \sqrt{t^5},$$
$$v(0,t) = 0, \quad v(1,t) = \sqrt{t^5},$$

where $f(x,t)$ and $g(x,t)$ are given by

$$f(x,t) = \frac{x\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\alpha)} t^{\frac{5}{2}-\alpha}, \qquad g(x,t) = \frac{x\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-\beta)} t^{\frac{5}{2}-\beta}.$$

Exact solution of this problem is $u(x,t) = v(x,t) = x\sqrt{t^5}$.

The maximum absolute errors are reported and compared between hybrid collection method and finite difference method in Tables 5 and 6.

Table 5: Compared errors between methods for Example 7.3 with $m = 5$, $\alpha = \beta = 0.3$, $L = 1$.

τ	Collection with TQ		Collection with FDM	
	$E_5^\infty(u)$	$E_5^\infty(v)$	$E_5^\infty(u)$	$E_5^\infty(v)$
0.03125	$1.48472839 \times 10^{-5}$	$3.05993758 \times 10^{-5}$	$7.79364362 \times 10^{-5}$	$7.79364362 \times 10^{-5}$
0.015625	$3.83997887 \times 10^{-6}$	$3.83997887 \times 10^{-6}$	$2.52338808 \times 10^{-5}$	$2.52338808 \times 10^{-5}$
0.0078125	$9.84732964 \times 10^{-7}$	$9.84732964 \times 10^{-7}$	$8.05476348 \times 10^{-6}$	$8.05476348 \times 10^{-6}$
0.00390625	2.5104961×10^{-7}	2.5104961×10^{-7}	$2.54753545 \times 10^{-6}$	$2.54753545 \times 10^{-6}$
0.001953125	6.3731242×10^{-8}	6.3731242×10^{-8}	$8.00626869 \times 10^{-7}$	$8.00626869 \times 10^{-7}$

Table 6: Compared errors between methods for Example 7.3 with $\tau = 1/1000$, $\alpha = \beta = 0.3$, $L = 1$.

m	Collection with TQ		Collection with FDM	
	$E_m^\infty(u)$	$E_m^\infty(v)$	$E_m^\infty(u)$	$E_m^\infty(v)$
5	$1.69048765 \times 10^{-8}$	$1.69048765 \times 10^{-8}$	$2.60661898 \times 10^{-7}$	$2.60661898 \times 10^{-7}$
6	$1.69047787 \times 10^{-8}$	$1.69047787 \times 10^{-8}$	$2.60654569 \times 10^{-7}$	$2.60654569 \times 10^{-7}$
7	$1.69042056 \times 10^{-8}$	$1.69042056 \times 10^{-8}$	$2.60644872 \times 10^{-7}$	$2.60644872 \times 10^{-7}$

Example 7.4. We consider the homogeneous equation which is time fractional coupled Burgers' equation,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial(uv)}{\partial x}, \tag{7.5}$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial(uv)}{\partial x}, \tag{7.6}$$

by initial conditions

$$u(x,0) = \sin(x), \qquad v(x,0) = \sin(x),$$

and the boundary conditions

$$u(0,t) = 0, \;\; u(1,t) = 0.84147E_\alpha(-t^\alpha),$$
$$v(0,t) = 0, \;\; v(1,t) = 0.84147E_\beta(-t^\beta),$$

where E_α is Mittag-Leffler function and it is given by $E_\alpha(x) = E(\alpha, x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+1)}$.

According to the proposed methods for $\alpha = \beta = 0.2$, the L_2 error for time fractional coupled Burgers' Eqs. (7.5), (7.6) is shown in Table 7.

Table 7: Compared errors between methods for Example 7.4 with $\alpha = \beta = 0.2$.

m	L_2–error by Collection with TQ		L_2–error by Collection with FDM	
	$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$
3	$1.33832026 \times 10^{-3}$	$1.33832026 \times 10^{-3}$	$2.84584171 \times 10^{-3}$	$2.84584171 \times 10^{-3}$
5	$7.00889205 \times 10^{-5}$	$7.00889205 \times 10^{-5}$	$1.16578069 \times 10^{-4}$	$1.16578069 \times 10^{-4}$
7	$5.80975101 \times 10^{-8}$	$5.80975101 \times 10^{-8}$	$1.24867303 \times 10^{-7}$	$1.24867303 \times 10^{-7}$
9	$1.98362640 \times 10^{-11}$	$1.98362640 \times 10^{-11}$	$1.0086298 \times 10^{-10}$	$1.0086298 \times 10^{-10}$
11	$2.19303051 \times 10^{-15}$	$2.19303051 \times 10^{-15}$	$1.80645883 \times 10^{-13}$	$1.80645883 \times 10^{-13}$

8. Conclusion

In this paper we presented a numerical method for solving the time-fractional Burgers' equation by utilizing the shifted Chebyshev polynomials and trapezoidal formula. Numerical results illustrate the validity and efficiency of the method and comparison between the maximum absolute errors of spectral collection method with trapezoidal formula and finite difference method shows the applicability and efficiency of the hybrid collection approach. Where, we see clearly that the error in the solution by trapezoidal formula is less than the error of the solution obtained by finite difference method.

Acknowledgment

The author thanks the anonymous reviewers for their careful reading of this manuscript and useful comments, which have helped with improving the presentation of the results.

References

[1] A. Atangana, *Derivative with two fractional orders: A new avenue of investigation toward revolution in fractional calculus*, Eur. Phys. J. Plus, **131** (2016), 373. 1

[2] A. Atangana, E. Alabaraoye, *Solving a system of fractional partial differential equations arising in the model of HIV infection of* $CD4^{+}$ *cells and attractor one-dimensional Keller-Segel equations*, Adv. Difference Equ., **2013** (2013), 14 pages. 1

[3] A. Atangana, D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Therm. Sci., **20** (2016), 763–769. 1

[4] A. Atangana, I. Koca, *Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order*, Chaos Solitons Fractals, **89** (2016), 447–454. 1

[5] H. Azizi, G. B. Loghmani, *A numerical method for space fractional diffusion equations using a semi-disrete scheme and Chebyshev collocation method*, J. Math. Comput. Sci., **8** (2014), 226–235. 1, 2.2

[6] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, edited by Richard von Mises and Theodore von Kármán, Advances in Applied Mechanics, Academic Press, Inc., New York, N. Y., (1948), 171–199. 1

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral methods in fluid dynamics*, Springer Series in Computational Physics, Springer-Verlag, New York, (1988). 3, 3

[8] L. Debnath, *Recent applications of fractional calculus to science and engineering*, Int. J. Math. Math. Sci., **2003** (2003), 3413–3442. 1

[9] M. Dehghan, A. Hamidi, M. Shakourifar, *The solution of coupled Burgers' equations using Adomian-Pade technique*, Appl. Math. Comput., **189** (2007), 1034–1047. 1

[10] M. Dehghan, J. Manafian, A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numer. Methods Partial Differential Equations, **26** (2010), 448–479. 1

[11] K. Diethelm, N. J. Ford, A. D. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations. Fractional order calculus and its applications*, Nonlinear Dynam., **29** (2002), 3–22. 4

[12] C. A. Fletcher, *Generating exact solutions of the twodimensional Burgers' equations*, Int. J. Numer. Meth. Fluids, **3** (1983), 213–216. 1

[13] R. Herrmann, *Numerical methods for fractional calculus*, Germany, (2011). 1

[14] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, (2000). 1

[15] A. Kadem, Y. Luchko, D. Baleanu, *Spectral method for solution of the fractional transport equation*, Rep. Math. Phys., **66** (2010), 103–115. 1

[16] S. Kazem, *An integral operational matrix based on Jacobi polynomials for solving fractional-order differential equations*, Appl. Math. Model., **37** (2013), 1126–1136. 1

[17] S. Kazem, S. Abbasbandy, S. Kumar, *Fractional-order Legendre functions for solving fractional-order differential equations*, Appl. Math. Model., **37** (2013), 5498–5510. 1

[18] S. Kazem, M. Shaban, J. Amani Rad, *Solution of the coupled Burgers equation based on operational matrices of d-dimensional orthogonal functions*, Z. Naturforsch., **67** (2012), 267–274. 1

[19] K. Krishnaveni, K. Kannan, S. Raja Balachandar, S. G. Venkatesh, *An efficient fractional polynomial method for space fractional diffusion equations*, Ain Shams Eng. J., 2016, (In press). 1

[20] S. Kumar, D. Kumar, S. Abbasbandy, M. M. Rashidi, *Analytical solution of fractional NavierStokes equation by using modified Laplace decomposition method*, Ain Shams Eng. J., **5** (2014), 569–574. 1

[21] C.-P. Li, F.-H. Zeng, *Numerical methods for fractional calculus*, CRC Press, China, (2015). 1, 5

[22] P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1985). 4

[23] F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*, Fractals and fractional calculus in continuum mechanics, Udine, (1996), 291–348, CISM Courses and Lect., Springer, Vienna, Springer, (1997). 1

[24] J.-P. Nee, J. Duan, *imit set of trajectories of the coupled viscous Burgers' equations*, Appl. Math. Lett., **11** (1998), 57–61. 1

[25] K. B. Oldham, J. Spanier, *The fractional calculus*, Theory and applications of differentiation and integration to arbitrary order, With an annotated chronological bibliography by Bertram Ross, Mathematics in Science and Engineering, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, (1974). 1

[26] I. Podlubny, *Fractional differential equations*, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, (1999). 1, 2.1

[27] S. S. Ray, *Exact solutions for time-fractional diffusion-wave equations by decomposition method*, Phys. Scripta, **75** (2007), 53–61. 1

[28] S. S. Ray, *A new approach for the application of Adomian decomposition method for the solution of fractional space diffusion equation with insulated ends*, Appl. Math. Comput., **202** (2008), 544–549. 1

[29] S. S. Ray, P. B. Poddar, R. K. Bera, *Analytical solution of a dynamic system containing fractional derivative of order one-half by Adomian decomposition method*, J. Appl. Mech., **72** (2005), 290–295. 1

[30] T. J. Rivlin, *Chebyshev polynomials*, From approximation theory to algebra and number theory, Second edition, Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, (1990). 2.4

[31] W. Rudin, *Principles of mathematical analysis*, Third edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Dsseldorf, (1976). 3

[32] A. Saadatmandi, M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl., **59** (2010), 1326–1336. 1

[33] M. Shahini, H. Adibi, *Rational approximations for solving differential equations of fractional order on semi-infinite interval*, Appl. Comput. Math., **13** (2014), 366–375. 1

[34] M. Shahini, H. Adibi, *New basis for solving fractional order models on semi-infinite intervals*, Appl. Math. Inf. Sci., **10** (2016), 1027–1033. 1

[35] Y.-X. Zeng, Y. Zeng, *Approximate solutions of the* Q*-discrete Burgers equation*, J. Math. Comput. Sci., **7** (2013), 241–248. 1