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Huang method for solving fully fuzzy linear system of equations

S.H. Nasseri [1],*, F. Zahmatkesh [1]

[1] *Department of Mathematics and Computer Sciences, Mazandaran University, P. O. Box 47415-1468, Babolsar, Iran.*

** E-mail: nasseri@umz.ac.ir*

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ABSTRACT

Recently, solving fuzzy versions of linear system of equations has been attracted many interests. As we know, unfortunately any practical method for finding a general solution of these systems is not at hand. In this paper, we concentrate on solving fully fuzzy linear system of equations and propose Huang method for computing a nonnegative solution of the fully fuzzy linear system of equations.

Keywords: Fuzzy arithmetic - Fully fuzzy linear system - Fuzzy number - Huang method

1. Introduction

Recently many authors have been considered a general model for fuzzy linear system of equations, that is fully fuzzy linear systems (FFLS) and have been proposed some methods for solving these systems [2,3,7]. In this paper, we focus on solving FFLSs which is defined as follows:

$$\tilde{A} \otimes \tilde{x} = \tilde{b},$$

where $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{x}^T = (\tilde{x}_1, \dots, \tilde{x}_n)_{1\times n}$and $\tilde{b}^T = (\tilde{b}_1, \dots, \tilde{b}_n)_{1\times n}$ such that $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{x}_j$ are triangular fuzzy numbers, for all $i, j = 1, \dots, n$ [2,3,7].

In Section 2, we first define FFLS and then discuss on solving FFLS. In Section 3, we propose an algorithm for solving them. We shall solve two FFLSs with the proposed method in Section 4. Finally, we conclude in Section 5.

2. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy sets theory (taken from [2,4,5]).

Definition 2.1. A fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is defined by its membership function:

$$\mu_{\tilde{a}} : \mathbb{R} \longrightarrow [0,1]$$

which assigns a real number $\mu_{\tilde{a}}$ in the interval [0, 1] , to each element $x \in \mathbb{R}$, where the value of $\mu_{\tilde{a}}$ at x shows the grade of membership of x in $\tilde{a}$.

Definition 2.2. A convex fuzzy set $\tilde{a}$ on $\mathbb{R}$ is a fuzzy number if the following conditions hold:
(a) Its membership function is piecewise continuous.
(b) There exist two intervals [a,b] and [b,c] such that $\mu_{\tilde{a}}$ is increasing on [a,b], decreasing on [b,c].

Definition 2.3. Let $\tilde{a} = (a^m, \alpha, \beta)$ denote the triangular fuzzy number, where $(a^m - \alpha, a^m + \beta)$ is support of $\tilde{a}$. The parameters α and β are respectively the left and right spreads.

Remark 2.1. We denote the set of all triangular fuzzy numbers by $F(\mathbb{R})$.
We now define arithmetic on triangular fuzzy numbers. Let $\tilde{a} = (a^m, \alpha, \beta)$ and $\tilde{b} = (b^m, \gamma, \theta)$ be two triangular fuzzy numbers. Define:

$$\begin{aligned} &x \geq 0\,, \quad x \in \mathbb{R}\,; \quad x\tilde{a} = (xa^m, x\alpha, x\beta),\\ &x < 0\,, \quad x \in \mathbb{R}\,; \quad x\tilde{a} = (xa^m, -x\beta, -x\alpha)\,,\\ &\tilde{a} + \tilde{b} = (a^m + b^m, \alpha + \gamma, \beta + \theta)\,,\\ &\tilde{a} - \tilde{b} = (a^m - b^m, \alpha - \theta, \beta - \gamma)\,. \end{aligned} \tag{2.1}$$

Remark 2.2. We show the zero triangular fuzzy number by $\tilde{0} = (0,0,0)$.

Definition 2.4. A fuzzy number $\tilde{a}$ is called positive (negative), denoted by $\tilde{a} > \tilde{0}$ ($\tilde{a} < \tilde{0}$), if its membership function $\mu_{\tilde{a}}(x)$ satisfies $\mu_{\tilde{a}}(x) = 0$, $\forall\, x \leq 0$ ($\forall\, x \geq 0$).

Definition 2.5. Two fuzzy numbers $\tilde{m} = (m, \alpha, \beta)$ and $\tilde{n} = (n, \gamma, \delta)$ are said to be equal, if and only if $m = n$, α = γ and β = δ.

Definition 2.6. For two fuzzy numbers $\tilde{m} = (m, \alpha, \beta)$ and $\tilde{n} = (n, \gamma, \delta)$ [4]:
If $\tilde{m} > \tilde{0}$ and $\tilde{n} > \tilde{0}$, then:

$$\tilde{m} \otimes \tilde{n} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, n\alpha + m\gamma, n\beta + m\delta) \tag{2.2}$$

Definition 2.7. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number [2].

A fuzzy matrix $\tilde{A}$ will be positive (negative) and denoted by $\tilde{A} > \tilde{0}$ ($\tilde{A} < \tilde{0}$), if each element of $\tilde{A}$ be positive (negative) number. Similarly non-negative and non-positive fuzzy matrices will be defined.

We follow the representation of the fuzzy matrix from [2] as following:
A fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, such that $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$, with the notation $\tilde{A} = (\mathrm{A}, \mathrm{M}, \mathrm{N})$, where $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are three $n \times n$ crisp matrices.

Definition 2.8. Consider the $m \times n$ fully fuzzy linear system of equations:

$$\begin{cases} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus \ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1,\\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus \ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2,\\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\ (\tilde{a}_{m1} \otimes \tilde{x}_1) \oplus \ldots \oplus (\tilde{a}_{mn} \otimes \tilde{x}_n) = \tilde{b}_m. \end{cases} \tag{2.3}$$

The matrix form of the above equations is

$$\tilde{A} \otimes \tilde{x} = \tilde{b},$$

where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$, is an $m \times n$ fuzzy matrix and $\tilde{x}_j, \tilde{b}_i \in F(\mathbb{R})$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. This system is called a fully fuzzy linear system (FFLS) (see in [2,6,7]).

In this paper, we are going to find a positive solution of FFLS $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A} = (\mathrm{A}, \mathrm{M}, \mathrm{N}) > \tilde{0}$, $\tilde{b} = (b, g, h) > \tilde{0}$, and $\tilde{x} = (x, y, z) > \tilde{0}$. So we have :

$$(A, M, N) \otimes (x, y, z) = (b, g, h).$$

Now if we assume that A is a nonsingular crisp matrix, then we can write:

$$(Ax, Ay + Mx, Az + Nx) = (b, g, h) \quad \Longrightarrow$$

$$\begin{cases} Ax = b,\\ Ay + Mx = g,\\ Az + Nx = h. \end{cases} \tag{2.4}$$

So,

$$\begin{cases} Ax = b & \Rightarrow x = A^{-1}b\,,\\ Ay = g - Mx & \Rightarrow y = A^{-1}g - A^{-1}Mx\,,\\ Az = h - Nx & \Rightarrow z = A^{-1}h - A^{-1}Nx\,. \end{cases}$$

Theorem 2.1. Let $\tilde{A} = (A, M, N) \geq \tilde{0}$, $\tilde{b} = (b, g, h) \geq \tilde{0}$ and the centric matrix A be inverse-nonnegative. Also, let $h \geq NA^{-1}b$, $g \geq MA^{-1}b$ and $(MA^{-1} + I)b \geq g$. Then, the system $\tilde{A} \otimes \tilde{x} = \tilde{b}$, has a positive fuzzy solution.
Proof . See [3].

3. Huang algorithm for solving linear system of equations

We know a class of algorithms, named the ABS class, for solving m linear equations in n unknowns with $m \leq n$ The mentioned class has been derived by analogy with the quasi-Newton methods. In exact arithmetic and under a certain condition (namely that the vectors $H_i a_i$ are nonzero) the algorithms are well defined and have the property that the iterate $\mathrm{m}+1$, with x_{m+1} solves the given equations. They are therefore algorithms of the direct type. One of the methods based upon the idea of solving at the i-th step the first i equations is Huang method.

Huang algorithm is defined by the following procedure (taken from [1]).

Algorithm 3.1. Huang Method

Step 1: Choose $x_1 \in \mathbb{R}^n$ be arbitrary, and $H_1 = I$. Set i=1.
Step 2: Compute $r_i = a_i^T x_i - b_i$ and $s_i = H_i a_i$.
Step 3: **If** $s_i = 0$ and $r_i = 0$, **then** set $x_{i+1} = x_i$, $H_{i+1} = H_i$ and go to 6 (the i-th equation is redundant). **If** $s_i = 0$ and $r_i \neq 0$, **then stop** (the i-th equation and hence the system is incompatible).
Step 4: Compute the search direction p_i by

$$p_i = H_i^T z_i, \tag{3.1}$$

with $z_i = a_i$ such that

$$z_i^T H_i a_i \neq 0. \tag{3.2}$$

Step5: Compute $\alpha_i = \frac{r_i}{a_i^T p_i}$, and let

$$x_{i+1} = x_i - \alpha_i H_i^T z_i. \tag{3.3}$$

Step 6: {Updating H_i } Update the matrix H_i by

$$H_{i+1} = H_i - H_i a_i w_i^T H_i, \tag{3.4}$$

where $w_i = \frac{a_i}{a_i^T H_i a_i}$ such that

$$w_i^T H_i a_i = 1.$$

(3.5)

Step 7: **If** $i = m$, **then stop** (x_{m+1} solves the system), **else** let $i = i + 1$ and go to step 2.

We note that after the completion of the algorithm, the general solution of system, if compatible, is written as follows:

$$x = x_{m+1} + H_{m+1}^T s, \tag{3.6}$$

where $s \in \mathbb{R}^n$ is arbitrary.

Below, we list some properties of the iterates H_i, p_i, x_i generated by the ABS class and omit the proofs (taken from [1] and [8]).

Theorem 3.1. Let $H_1 \in \mathbb{R}^{n \times n}$ be a nonsingular matrix and let a_1, ..., a_m be linearly independent vectors in $\mathbb{R}^n$ $(m \leq n)$. Consider, for $i = 1, \dots, m$, the sequence of matrices H_i generated by update (3.4), with w_i arbitrary vector in $\mathbb{R}^n$. Then for $i \leq j \leq m$, the vectors $H_i a_j$ are nonzero and linearly independent.

Corollary 3.1. The vector $H_i a_i$ computed by the ABS algorithm is zero only if a_i is a linear combination of a_1, ..., a_{i-1}.

Theorem 3.2. Given m arbitrary vectors a_1, ..., $a_m \in \mathbb{R}^n$ and an arbitrary non-singular matrix $H_1 \in \mathbb{R}^{n \times n}$, consider the sequence of matrices H_2, ..., H_{m+1} generated by (3.4) with w_i satisfying (3.5), if such a w_i exists, otherwise according to $H_{i+1} = H_i$. Then the following relations are true for $i = 2, \dots, m + 1$:

$$H_i a_j = 0 , \quad j \leq i - 1.$$

Corollary 3.2. The vector $H_i a_i$ computed by the ABS algorithm is zero if and only if a_i is a linear combination of a_1, ..., a_{i-1}.

Corollary 3.3. The ABS algorithm is well defined for arbitrary rank of A.

Theorem 3.3. For $1 \le i \le m+1$ the rank of the matrices H_i, defined by update (3.4) with H_1 nonsingular and w_i satisfying (3.5), is equal to $n-i+1$.

Theorem 3.4. Let H_1 be symmetric positive definite and consider the matrices H_i generated by (3.4) with w_i some vector. Then the following statement are true for $i = 1, \dots, m$.

(a) The following formula for $w_i$is well defined and satisfies (3.5):

$$w_i = \frac{a_i}{a_i^T H_i a_i}. \qquad (3.7)$$

(b) The denominator in (3.7) is strictly positive.
(c) The matrix H_{i+1} is symmetric.

Remark 3.1. Equation (3.7) for w_i was used by Huang, with $H_1 = I$ and $z_i = a_i$, in the paper which led to the development of the ABS class (see in [1]).

4. Numerical Examples

In this section, we illustrate the proposed method with solving two fully fuzzy linear systems of equations.

Example 4.1. Consider the following FFLS:

$$\begin{pmatrix} (19,1,1) & (12,1.5,1.5) & (6,0.5,0.2) \\ (2,0.1,0.1) & (4,0.1,0.4) & (1.5,0.2,0.2) \\ (2,0.1,0.2) & (2,0.1,0.3) & (4.5,0.1,0.1) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (1897,427.7,536.2) \\ (434.5,76.2,109.3) \\ (535.5,88.3,131.9) \end{pmatrix}.$$

The solution is obtained by Huang algorithm as follows:

$$\tilde{x} = (37,7,13.3016), \quad \tilde{y} = (62,5.5,4.5794), \quad \tilde{z} = (75,10.2,13.9196).$$

Example 4.2. Consider the following FFLS:

$$\begin{pmatrix} (9,0.2,0.2) & (1,0.4,0.3) & (3,0.3,0.4) & (4,0.2,0.1) \\ (2,0.3,0.1) & (7,0.4,0.2) & (2,0.2,0.3) & (1,0.1,0.3) \\ (1,0.3,0.2) & (1,0.5,0.2) & (6,0.3,0.1) & (3,0.2,0.3) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} (27.500,9.300,19.125) \\ (17.750,9.025,8.525) \\ (13.250,6.900,12.725) \end{pmatrix}.$$

The solution is obtained by Huang algorithm as follows:

$$\tilde{x} = (1.9974,0.4114,1.0596), \qquad \tilde{y} = (1.5006,0.7722,0.2600),$$
$$\tilde{z} = (0.9960,0.3628,1.1367), \qquad \tilde{t} = (1.2586,0.5465,1.1362).$$

5. Conclusion

In this paper, we used a certain case of the ABS class of algorithms for solving fully fuzzy linear systems. We examined Algorithm 3.1 by solving two fully fuzzy linear systems.

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