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A sufficient condition for coinciding the Green graphs of semigroups

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Abstract

A necessary condition for coinciding the Green graphs $\Gamma_{\mathcal{L}}(S), \Gamma_{\mathcal{R}}(S), \Gamma_{\mathcal{J}}(S), \Gamma_{\mathcal{D}}(S)$ and $\Gamma_{\mathcal{H}}(S)$ of a finite semigroup S has been studied by Gharibkhajeh [A. Gharibkhajeh, H. Dosstie, Bull. Iranian Math. Soc., **40** (2014), 413–421]. Gharibkhajeh et al. proved that the coinciding of Green graphs of a finite semigroup S implies the regularity of S. However, the converse is not true because of certain well-known examples of finite regular semigroups. We look for a sufficient condition on non-group semigroups that implies the coinciding of the Green graphs. Indeed, in this paper we prove that for every non-group quasi-commutative finite semigroup, all of the Green graphs are isomorphic.

Keywords: Quasi-commutativity, finitely presented semigroups, Green relations, Green graphs.
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1. Introduction

Let S be a finite semigroup. Following the notation of [4], the left Green Graph $\Gamma_{\mathcal{L}}(S)$ is an undirected graph with vertices $\mathcal{L}_i, (1 \leqslant i \leqslant t)$ where the $\mathcal{L}_i$s are the left Green classes of the semigroup S and two vertices $\mathcal{L}_i$, $\mathcal{L}_j$ are adjacent in $\Gamma_{\mathcal{L}}(S)$ if and only if $\gcd(|\mathcal{L}_i|, |\mathcal{L}_j|) > 1$. These graphs are indeed the generalization of the conjugacy graphs of finite groups studied by Adan-Bante [1]. The right Green graph $\Gamma_{\mathcal{R}}(S)$, the intersection Green graph $\Gamma_{\mathcal{H}}(S)$, the join Green graph $\Gamma_{\mathcal{D}}(S)$, and finally the $\mathcal{J}$-classes Green graph $\Gamma_{\mathcal{J}}(S)$ are defined in a similar way. Investigating these graphs is of interest because of their ability in identifying certain types of finite semigroups, the non-group non-regular quasi-commutative semigroups. As usual, an associative algebraic structure $(S, \cdot)$ is called quasi-commutative if, for every elements $a, b \in S$, there exists a positive integer r such that $ab = b^r a$. For useful information on quasi-commutative semigroups and examples, one may see [2, 3, 5–7]. Our main results on this type of semigroup are the following:

Proposition A. *For every non-commutative quasi-commutative semigroup* S*, all Green graphs are isomorphic.*

Proposition B. *If* b *is a non idempotent element of a nowhere commutative quasi-commutative finite semigroup* S*, then* b *is regular if and only if* $|[b]_{\mathcal{J}}| > 1$.

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Proposition C. *Let S be a finite non-regular nowhere commutative quasi-commutative semigroup. Then all of the Green graphs of S are isomorphic to $nK_1 \cup K_m$ where m is the number of $\mathcal{L}$-classes and n is the number of non-regular non idempotent elements of S. Moreover, these graphs are not complete.*

2. The proofs

We start the proofs with a key lemma.

Lemma 2.1. *Every non idempotent regular element b of a finite semigroup S satisfies $|[b]_{\mathcal{J}}| > 1$.*

Proof. For a regular element $b \in S$, we may use a method of proof similar to the proof of Lemma 1.14. of [3]. Indeed, there exists an element $x \in S$ such that $b = bxb$, and then $y = xbx$ is an inverse for b. This yields $yby = (xbx)b(xbx) = x(bxb)(xbx) = x(bxb)x = xbx = y$ and $byb = b(xbx)b = (bxb)xb = bxb = b$. Let $y \neq b$. So, $yby = y$ and $byb = b$ implies that $y \in [b]_{\mathcal{J}}$ and therefore $|[b]_{\mathcal{J}}| > 1$. If $y = b$ so $b^3 = b$ and we have the following relations:

$$b = b \cdot b^2 \cdot b^2,\ b^2 = b \cdot b \cdot b^2,$$

which shows that $b^2 \in [b]_{\mathcal{J}}$ and so $|[b]_{\mathcal{J}}| > 1$. □

Proof of Proposition A. We consider the different cases as follows:

Case 1. $x\mathcal{L}y \implies x\mathcal{R}y$. If $x\mathcal{L}y$ so, $y = xu$ and $x = yv$, for some $u, v \in S$. Since S is quasi-commutative then there exist integers r_u, r_v such that $xu = u^{r_u}x$ and $yv = v^{r_v}y$, respectively. Therefore, the identities $y = u_1x$ and $x = v_1y$ show that $x\mathcal{R}y$, where $u_1 = u^{r_u}$ and $v_1 = v^{r_v}$.

Case 2. $x\mathcal{R}y \implies x\mathcal{L}y$. In a similar way to the first case and considering the definition of the right Green graphs.

Case 3. $x\mathcal{L}y \iff x\mathcal{H}y$. As in Case 1, $x\mathcal{L}y$ yields $x\mathcal{R}y$. So, by the definition of $\mathcal{H}$-relation, we get $x\mathcal{H}y$. The converse is obvious.

Case 4. $x\mathcal{R}y \iff x\mathcal{H}y$. Similar to Cases 2 and 3.

Case 5. $x\mathcal{L}y \implies x\mathcal{J}y$. If $x\mathcal{L}y$ then there exist $u, v \in S$ such that $y = xu$ and $x = yv$. Due to the quasi-commutativity of S, there exist positive integers r_v, r_u, r_y, r_x such that

$$yv = v^{r_v}y,\ xu = u^{r_u}x,\ vy = y^{r_y}v,\ ux = x^{r_x}u.$$

There are three cases to consider:

(1) $r_v > 1, r_y > 1$. We get:

$$x = yv = v^{r_v}y = v^{r_v-1}(vy) = v^{r_v-1}(y^{r_y}v) = \left(v^{r_v-1}\right) y \left(y^{r_y-1}v\right),$$

which yields $x = u_1yv_1$, $(u_1 = v^{r_v-1}, v_1 = y^{r_y-1}v)$.

(2) $r_v = 1, r_y \geqslant 1$. We get:

$$x = yv = vy = v(xu) = v(yv)u = u_2yv_2,\ (u_2 = v, v_2 = vu).$$

(3) $r_v > 1, r_y = 1$. In this situation, we have:

$$x = yv = v^{r_v}y = v^{r_v-1}(vy) = v^{r_v-1}(yv),$$

which yields $x = u_3yv_3$, $(u_3 = v^{r_v-1}, v_3 = v)$. The proof of $y = u_ixv_j$ for some $u_i, v_j \in S$ is similar.

Case 6. $x\mathcal{R}y \implies x\mathcal{J}y$. Clearly, $x\mathcal{R}y$ yields $x\mathcal{L}y$ so, $x\mathcal{J}y$.

Case 7. $x\mathcal{J}y \Longrightarrow x\mathcal{L}y$. $x\mathcal{J}y$ implies that $x = u_1yv_1$ and $y = u_2xv_2$ for some u_1, u_2, v_1 and v_2 in S. Because of the quasi-commutativity of S, we have $x = (y^{r_y}u_1)v_1 = yu_2$, $(u_2 = y^{r_y-1}u_1v_1)$ and $y = (x^{r_x}u_2)v_2 = xv_3$, $(v_3 = x^{r_x-1}u_2v_2)$ where r_y and r_x are both positive integers. This shows that $x\mathcal{L}y$.

Case 8. $x\mathcal{D}y \Longrightarrow x\mathcal{J}y$. Since $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{H}$, then $\mathcal{D} \subseteq \mathcal{L}$. So, the proof is obvious.

Case 9. $x\mathcal{J}y \Longrightarrow x\mathcal{D}y$. Let $x\mathcal{J}y$. Then, there are elements u_1, u_2, v_1 and v_2 in S such that

$$x = u_1yv_1, \ y = u_2xv_2.$$

Setting $z = u_1y, k = v_1$ yields $x = u_1yv_1 = zk$. So, $z = u_1y = u_1(u_2xv_2)$. By the quasi-commutativity of S, there are integers r_1, r_2 such that $u_2x = x^{r_1}u_2, u_1x = x^{r_2}u_1$. Therefore,

$$z = u_1(u_2xv_2) = u_1\,(x^{r_1}u_2)\,v_2 = (u_1x)\left(x^{r_1-1}u_2v_2\right) = xu_3,$$

where, $u_3 = \left(x^{r_2-1}u_1\right)\left(x^{r_1-1}u_2v_2\right)$. This shows that $x\mathcal{L}z$. Moreover, $y = u_2xv_2 = u_2(zv_3)$ where, $v_3 = v_1v_2$ and so there is an integer $r_{v_3} \geqslant 1$ such that

$$y = u_2xv_2 = v_4z, (v_4 = u_2v_3^{r_{v_3}}).$$

The latter identity and $z = u_1y$ confirm that $z\mathcal{R}y$. This completes the proof of $x\mathcal{D}y$. □

Proof of Proposition B. Let $x\mathcal{J}b$ where $x \in S$ and $x \neq b$. So, there exist elements $u_i, v_i \in S, (i = 1, 2)$ such that

$$b = u_1xv_1, \ x = u_2bv_2.$$

So we have $b = u_1(u_2bv_2)v_1 = u_3bv_3$ where $u_3 = u_1u_2, v_3 = v_2v_1$. Because of the quasi-commutativity of S, we can find positive integers r_b such that $u_3b = b^{r_b}u_3$ and therefore $b = b^{r_b}y$ where $y = u_3v_3$. Considering two different cases for r_b, we have:

(1) If $r_b > 1$ so $b = b^{r_b}y = b^{r_b-1}(by)$ and by quasi-commutativity of S we have $b = b^{r_b-1}y^{r_y}b$ where r_y is some positive integer.

(2) If $r_b = 1$ then $u_3b = bu_3$ and so the nowhere commutativity of the semigroup gives $u_3 = b$.

Therefore by the quasi-commutativity of S we have:

$$b = u_3bv_3 = b \cdot (v_3^{r_{v_3}}b) = b \cdot v_4 \cdot b, (v_4 = v_3^{r_{v_3}}),$$

where r_{v_3} is a positive integer. This means that b is a regular element of S. For the converse, we consider Lemma 2.1. □

Proof of Proposition C. By using Proposition A, we get that

$$\Gamma_{\mathcal{L}}(S) \cong \Gamma_{\mathcal{R}}(S) \cong \Gamma_{\mathcal{J}}(S) \cong \Gamma_{\mathcal{D}}(S) \cong \Gamma_{\mathcal{H}}(S).$$

So, identifying the Green graph of S one needs only to consider the $\mathcal{L}$-classes of S. If there are n non-regular elements $b_1, b_2, \cdots, b_n \in S$ then by a consequence of Proposition B, we get:

$$nK_1 = \bigcup_{1}^{n} \Gamma_{\mathcal{L}}([b_i]).$$

By considering the set of all $\mathcal{L}$-classes of S as $\{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_m\}$, where each class contains at least two elements, we construct the sub-graph K_m of $\Gamma_{\mathcal{L}}(S)$. Consequently,

$$\Gamma_{\mathcal{L}}(S) \cong nK_1 \cup K_m.$$

Since S is non-regular, $\Gamma_{\mathcal{L}}(S)$ is not a complete graph. □

Conclusion 2.2. Using a similar proof, we may extend Proposition A for quasi-hamiltonian semigroups. By definition, the semigroup S is quasi-hamiltonian if and only if for every elements $a, b \in S$ there are positive integers r_a, r_b such that $ab = b^{r_b}a^{r_a}$.

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