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Duality and biorthogonality for g-frames in Hilbert spaces

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Abstract

The main aim of this paper is to define the generalized Riesz-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases. We characterize exactly properties of the first sequence in terms of the associated one, which yields duality relations for the abstract g-frame setting.

Keywords: g-orthonormal basis, g-frames, g-Riesz-dual sequence, Riesz-duality.
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1. Introduction

Duality principles in Gabor theory such as the Ron-Shen duality principle [13] and the Wexler-Raz biorthogonality relations [17] play a fundamental role for analyzing Gabor systems. Casazza et al. in [4] introduced a general approach to derive duality principles in abstract frame theory. For each sequence in a separable Hilbert space they defined a Riesz-dual sequence dependent only on two orthonormal bases. They characterize exactly properties of the first sequence in terms of the Riesz-dual sequence, which yields duality relations for the frame setting. Frames were first introduced by Duffin and Schaeffer [9] in the context of nonharmonic Fourier series and reintroduced in 1986 by Daubechies et al. in [8]. Currently, frames play important roles in many applications in mathematics, science, and engineering such as signal processing, image processing, data compression, etc.

Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ be orthonormal bases for a separable Hilbert space $\mathcal{H}$ and let $f = \{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty$ for all $j \in I$. Then the Riesz-dual sequence (R-dual sequence) of $\{f_i\}_{i\in I}$ with respect to $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ as the sequence $\{\mathscr{W}_j^f\}_{j\in I}$ is given by:

$$\mathscr{W}_j^f = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \quad \forall j \in I.$$

This simple construction gives a powerful tool for deriving duality principles in general frame theory. There exists a symmetric relation between the sequences $\{\mathscr{W}_j^f\}_{j\in I}$ and $\{f_i\}_{i\in I}$ as follows:

$$f_i = \sum_{j\in I} \langle \mathscr{W}_j^f, h_i\rangle e_j, \quad \forall i \in I.$$

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In particular, this shows that $\{f_i\}_{i\in I}$ is the R-dual sequence for $\{\mathscr{W}_j^f\}_{j\in I}$ with respect to $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$. We refer the reader to the articles [6, 7, 14, 18] for an introduction about the theory and applications of R-dual sequences.

Recently, Sun in [15, 16] and Casazza and Kutyniok in [3] introduced a generalization of frames which covers many other recent generalizations of frames, e.g., bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames, and a class of time-frequency localization operators. Sun showed that all of the above applications of frames are special cases of generalized frames.

Let $\mathcal{H}$ and $\mathcal{K}$ be two separable Hilbert spaces and let $\{V_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{K}$ and $B(\mathcal{H}, V_i)$ denote the collection of all bounded linear operators from $\mathcal{H}$ into V_i for all $i \in I$. Then, $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) :\ i \in I\}$ is a generalized frame or simply a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if there exist constants $0 < C \leqslant D < \infty$ such that:

$$C\|f\|^2 \leqslant \sum_{i\in I} \|\Lambda_i f\|^2 \leqslant D\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants C and D are called g-frame bounds. If only the right-hand inequality of (1.1) is required, we call it a g-Bessel sequence. Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following examples.

Example 1.1. Let $\mathcal{H} = \mathbb{C}^n$ and $V_1 = V_2 = \ldots = V_n = \mathbb{C}^{n+1}$. Define

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \ \Lambda_2 = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \ \ldots, \ \Lambda_n = \begin{bmatrix} 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Then, the set $\Lambda = \{\Lambda_i\}_{i=1}^n$ is a g-frame for $\mathbb{C}^n$ with respect to $\mathbb{C}^{n+1}$ with g-frame bounds $A = 2$ and $B = n+1$. To see this explicitly, note that for any $f = (z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$, we have

$$\sum_{i=1}^n \|\Lambda_i f\|^2 = 2|z_1|^2 + 3|z_2|^2 + \ldots + (n+1)|z_n|^2.$$

From this, we have

$$2\|f\|^2 \leqslant \sum_{i=1}^n \|\Lambda_i f\|^2 \leqslant (n+1)\|f\|^2.$$

In frames theory an input signal is represented by a collection of scalar coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory equals $\ell^2(I)$. However, in g-frames theory an input signal is represented by a collection of vector coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

$$\Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} = \Big\{\{g_i'\}_{i\in I} \big|\ g_i' \in V_i,\ \sum_{i\in I} \|g_i'\|^2 < \infty\Big\}.$$

In order to analyze a signal $f \in \mathcal{H}$, i.e., to map it into the representation space, the analysis operator $T_\Lambda : \mathcal{H} \to \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2}$ given by $T_\Lambda f = \{\Lambda_i f\}_{i\in I}$ is applied. The associated synthesis operator, which provides a mapping from the representation space to $\mathcal{H}$, is defined to be the adjoint operator $T_\Lambda^* : \big(\sum_{i\in I} \oplus V_i\big)_{\ell^2} \to \mathcal{H}$, which is given by $T_\Lambda^*(\{g_i'\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i'$. By composing T_Λ and T_Λ^* we obtain the g-frame operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$, $S_\Lambda f = T_\Lambda^* T_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$, which is a positive, self-adjoint and invertible operator and

$C \leqslant \|S_\Lambda\| \leqslant D$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widehat{\Lambda}_i\}_{i\in I}$ where $\widehat{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ which is also a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with $\frac{1}{D}$ and $\frac{1}{C}$ as its lower and upper frame bounds, respectively. Also we have

$$f = \sum_{i\in I} \Lambda_i^* \widehat{\Lambda}_i f = \sum_{i\in I} \widehat{\Lambda}_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}.$$

Moreover, $\{\Lambda_i S_\Lambda^{-\frac{1}{2}}\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$.

Generalized Riesz-dual sequence or simply g-R-dual sequence is a natural generalization of R-dual sequence which provides a powerful tool in the analysis of duality relations in general g-frame theory. The purpose of this paper is to introduce the concept of Riesz-dual sequence for g-frames. We give characterizations of g-R-dual sequences and prove that g-R-dual sequences share many useful properties with R-dual sequences. In this article, we show that in fact for each sequence of operators we can construct a corresponding sequence of operators with a kind of duality relation between them. This construction is used to prove duality principles in g-frame theory, which can be regarded as general versions of several well-known duality principles for g-frames. We also give a generalized version of Riesz-dual sequences.

The content of this paper is as follows: In the rest of this section we will briefly recall the necessary parts from g-bases, g-orthonormal bases, and g-Riesz bases. For more information we refer to [1, 2, 5, 10, 11]. In Section 2, we define the g-R-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases as generalization of Riesz-dual sequence. In this section, we characterize to which extent the g-R-dual sequence of a g-Bessel sequence depends on the chosen g-orthonormal bases. In Section 3, first we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent). Finally, Section 4 deals with duality principle for g-frames. In this section we study properties of dual g-frames and canonical dual g-frames.

Definition 1.2. A generalized Schauder basis or simply a g-basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ is a family of onto operators $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j)|\ j \in I\}$ such that for all $f \in \mathcal{H}$ there exist unique vectors $g_j \in W_j, i \in I$ with

$$f = \sum_{j\in I} \Gamma_j^* g_j. \tag{1.2}$$

In this case, there exist unique operators $\Lambda_j \in B(\mathcal{H}, W_j)$ such that

$$f = \sum_{j\in I} \Gamma_j^* \Lambda_j f = \sum_{j\in I} \Lambda_j^* \Gamma_j f,$$

for all $f \in \mathcal{H}$. Moreover, the sequences $\{\Gamma_j\}_{j\in I}$ and $\{\Lambda_j\}_{j\in I}$ are g-biorthogonal, i.e., $\Lambda_i \Gamma_j^* g_j = \delta_{ij} g_j$ for all $i, j \in I, g_j \in W_j$ and $\{\Lambda_j\}_{j\in I}$ itself forms a g-basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ that so-called dual g-basis of $\{\Gamma_j\}_{j\in I}$. A g-basis is an unconditional g-basis, if the series in (1.2) converges unconditionally. Consequently, for a g-basis the ordering in (1.2) can be crucial. If $\{\Lambda_i\}_{i\in I}$ is a g-basis only for its closed linear span, we call it a g-basic sequence with respect to $\{W_i\}_{i\in I}$.

Definition 1.3. Let $\{\Xi_i \in B(\mathcal{H}, W_i)|\ i \in I\}$ be a sequence of operators. Then

(i) $\{\Xi_i\}_{i\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if $\mathcal{H} = \overline{\operatorname{span}}\{\Xi_i^*(W_i)\}_{i\in I}$.

(ii) $\{\Xi_i\}_{i\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, if $\Xi_i \Xi_j^* = \delta_{ij} I_{W_j}$ for all $i, j \in I$.

(iii) A g-complete and g-orthonormal system $\{\Xi_i\}_{i\in I}$ is called a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

Definition 1.4. A sequence $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j)|\ j \in I\}$ is called a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, if $\{\Gamma_j\}_{j\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and there exist constants $0 < A \leqslant B < \infty$ such that

$$A \sum_{j\in I} \|g_j\|^2 \leqslant \Big\|\sum_{j\in I} \Gamma_j^* g_j\Big\|^2 \leqslant B \sum_{j\in I} \|g_j\|^2, \tag{1.3}$$

for all sequences $\{g_j\}_{j\in I} \in \big(\sum_{j\in I} \oplus W_j\big)_{\ell^2}$. We define the g-Riesz basis bounds for $\{\Gamma_j\}_{j\in I}$ to be the largest number A and the smallest number B such that this inequality (1.3) holds. If $\{\Gamma_j\}_{j\in I}$ is a g-Riesz basis only for $\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}$, we call it a g-Riesz basic sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$.

The following well-known characterization of g-orthonormal bases is sometimes more useful which is taken from [2].

Lemma 1.5. *Let $\Xi = \{\Xi_i\}_{i\in I}$ be a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then the following conditions are equivalent:*

(i) *Ξ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*
(ii) $\sum_{i\in I} \Xi_i^*\Xi_i = I_{\mathcal{H}}$.
(iii) $\|f\|^2 = \sum_{i\in I} \|\Xi_i^*\Xi_i f\|^2, \quad \forall f \in \mathcal{H}$.
(iv) $\|f\|^2 = \sum_{i\in I} \|\Xi_i f\|^2, \quad \forall f \in \mathcal{H}$.
(v) $< f, g >= \sum_{i\in I} < \Xi_i f, \Xi_i g >, \quad \forall f, g \in \mathcal{H}$.
(vi) *If $\Xi_i f = 0$ for all $i \in I$, then $f = 0$.*

For any given g-frame there is a natural procedure to construct a g-Riesz basis with the same g-frame bounds, see, e.g., [1] for a proof of this standard result.

Lemma 1.6. *Let $\{\Xi_j\}_{j\in I}$ be a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $U : \mathcal{H} \to \mathcal{H}$ a bounded bijective operator. Then the following items hold.*

(i) *The sequence $\{\Xi_j U^*\}_{j\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ with g-frame operator UU^* and optimal bounds $\frac{1}{\|U^{-1}\|^2}, \|U\|^2$.*
(ii) *The dual g-Riesz basis of $\{\Xi_j U^*\}_{j\in I}$ is $\{\Xi_j U^{-1}\}_{j\in I}$ with g-frame operator $(UU^*)^{-1}$ and the optimal bounds are $\frac{1}{\|U\|^2}, \|U^{-1}\|^2$.*
(iii) *Let $\Gamma = \{\Gamma_j\}_{j\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ with optimal bounds A, B. Then $\{\Xi_j S_\Gamma^{\frac{1}{2}}\}_{j\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ with optimal bounds A, B. The dual g-Riesz basis of $\{\Xi_j S_\Gamma^{\frac{1}{2}}\}_{j\in I}$ is $\{\Xi_j S_\Gamma^{-\frac{1}{2}}\}_{j\in I}$, with optimal bounds $\frac{1}{B}, \frac{1}{A}$.*
(iv) *Let $\Gamma = \{\Gamma_j\}_{j\in I}$ be a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, then $\{\Gamma_j S_\Gamma^{-\frac{1}{2}}\}_{j\in I}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$.*
(v) *Let $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j) \mid j \in I\}$ be arbitrary sequence. If $\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I} = \mathcal{H}$ and*

$$\Big\|\sum_{j\in I} \Gamma_j^* g_j\Big\|^2 = \sum_{j\in I} \|g_j\|^2, \quad \forall \{g_j\}_{j\in I} \in \Big(\sum_{j\in I} \oplus W_j\Big)_{\ell^2},$$

then $\Gamma = \{\Gamma_j\}_{j\in I}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

Let $\Xi = \{\Xi_i\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. If $f = \sum_{i\in I} \Xi_i^* g_i$, then the coordinate representation of $f \in \mathcal{H}$ relative to the g-orthonormal basis Ξ is $[f]_\Xi = \{g_i\}_{i\in I}$. In this case $\{g_i\}_{i\in I} \in \big(\sum_{i\in I} \oplus W_i\big)_{\ell^2}$ and $\|f\| = \big\|[f]_\Xi\big\|_{\ell^2}$.

Definition 1.7. Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Xi' = \{\Xi_i'\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. The transition matrix from Ξ to Ξ' is the matrix $B = [B_{ij}]$ whose (i,j)-entry is $B_{ij} = \Xi_i'\Xi_j^*$ for all $i, j \in I$. We also have $B[f]_\Xi = [f]_{\Xi'}$ where, $[f]_\Xi$ and $[f]_{\Xi'}$ are the coordinate representation of an arbitrary vector $f \in \mathcal{H}$ in the basis Ξ and Ξ', respectively. We show that the transition matrix from Ξ' to Ξ is $B^{-1} = B^*$. Let $B^* = [B_{ij}^*]$, then $B_{ij}^* = (B_{ji})^* = \Xi_i\Xi_j'^*$ for all $i, j \in I$. By Lemma 1.5 we have

$$[BB^*]_{ij} = \sum_{k\in I} B_{ik}B_{kj}^* = \sum_{k\in I} E'_i E_k^* E_k E_j'^* = E_i'\Big(\sum_{k\in I} E_k^* E_k\Big)E_j'^* = E'_i I_{\mathcal{H}} E_j'^* = E'_i E_j'^* = \delta_{ij} I_{W_j}.$$

Similarly, $[B^*B]_{ij} = \delta_{ij} I_{W_j}$. This implies that $BB^* = B^*B = I$, where I is the identity matrix.

Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following example.

Example 1.8. Let $\mathcal{H} = \mathbb{C}^{2n}$ and $W_1 = W_2 = \ldots = W_n = \mathbb{C}^2$. Define

$$\Xi_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Xi_n = \begin{bmatrix} 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

A direct calculation shows that $\|\Xi_k\| = 1$ and $\Xi_k \Xi_\ell^* = \delta_{k\ell}$ for any $1 \leqslant k, \ell \leqslant n$. We also have

$$\sum_{k=1}^{n} \|\Xi_k f\|^2 = \sum_{k=1}^{n} (|z_{2k-1}|^2 + |z_{2k}|^2) = \|f\|^2, \quad \forall f = \{z_i\}_{i=1}^{2n} \in \mathbb{C}^{2n}.$$

Therefore $\Xi = \{\Xi_k\}_{k=1}^n$ is a g-orthonormal basis for $\mathbb{C}^{2n}$ with respect to $\mathbb{C}^2$. Similarly, the sequence $\Psi = \{\Psi_k\}_{k=1}^n$ defined by

$$\Psi_1 = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Psi_n = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix},$$

is also a g-orthonormal basis for $\mathbb{C}^{2n}$ with respect to $\mathbb{C}^2$ and the matrix

$$B = [\Psi_i \Xi_j^*]_{n \times n} = \begin{bmatrix} A & & \bar{0} \\ & \ddots & \\ \bar{0} & & A \end{bmatrix},$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the transition matrix from Ξ to Ψ. Hence, for any $f \in \mathbb{C}^{2n}$ we have $B[f]_\Xi = [f]_\Psi$.

Example 1.9. Let $\mathcal{H} = \mathbb{C}^{2n}$ and $W_1 = W_2 = \ldots = W_{2n} = \mathbb{C}^2$. Define

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 2 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Gamma_n = \begin{bmatrix} 0 & 0 & \ldots & 2n-1 & 0 \\ 0 & 0 & \ldots & 0 & 2n \end{bmatrix}.$$

Since, for every $g_i = (z_{2i-1}, z_{2i}) \in \mathbb{C}^2$, we have $\left\| \sum_{i=1}^n \Gamma_i^* g_i \right\|^2 = \sum_{i=1}^{2n} i^2 |z_i|^2$. Thus $\{\Gamma_i\}_{i=1}^n$ is a g-Riesz basis for $\mathbb{C}^{2n}$ with respect to $\mathbb{C}^2$ with g-Riesz bounds 1 and $4n^2$. Moreover, we can write $\{\Gamma_i\}_{i=1}^n = \{\Xi_i U^*\}_{i=1}^n$, where U is a bounded bijective operator defined by

$$U = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 2n \end{bmatrix},$$

and $\Xi = \{\Xi_k\}_{k=1}^n$ is the g-orthonormal basis defined in Example 1.8.

2. The g-R-dual sequence

In this section we define the g-R-dual sequence from a sequence of operators. Then we exactly characterize to which extent the g-R-dual sequence of a g-Bessel sequence depends on the chosen g-orthonormal bases.

Definition 2.1. Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. Let $\Lambda = \{\Lambda_i : \mathcal{H} \to V_i |\ i \in I\}$ be such that the series $\sum_{i\in I} \Lambda_i^* g_i'$ is convergent for all $\{g_i'\}_{i\in I} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}$. For all $j \in I$, let

$$\Gamma_j^\Lambda : \mathcal{H} \to W_j, \quad \Gamma_j^\Lambda = \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i.$$

Then $\{\Gamma_j^\Lambda\}_{j\in I}$ is called the generalized Riesz-dual sequence (g-R-dual sequence) for the sequence Λ with respect to (Ξ, Ψ).

Notice that the hypothesis that the series $\sum_{i\in I}\Lambda_i^* g_i'$ is convergent for all $\{g_i'\}_{i\in I}\in\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ is always fulfilled if the sequence $\Lambda=\{\Lambda_i\}_{i\in I}$ is g-Bessel sequence with respect to $\{V_i\}_{i\in I}$.

Example 2.2. Let $\mathcal{H}=\mathbb{C}^{2n}$ and let $\{\Xi_i\}_{i=1}^n$, $\{\Psi_i\}_{i=1}^n$ be the g-orthonormal bases for $\mathcal{H}$ with respect to $\mathbb{C}^2$ defined in Example 1.8. Define

$$\Lambda_1=\begin{bmatrix}1&1&\dots&0&0\\0&1&\dots&0&0\end{bmatrix},\dots,\Lambda_n=\begin{bmatrix}0&0&\dots&1&1\\0&0&\dots&0&1\end{bmatrix}.$$

Then, $\Lambda=\{\Lambda_i\}_{i=1}^n$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B=3$. The g-R-dual sequence for the sequence Λ with respect to (Ξ,Ψ) is defined as follows:

$$\Gamma_1^\Lambda=\begin{bmatrix}0&1&\dots&0&0\\1&1&\dots&0&0\end{bmatrix},\dots,\Gamma_n^\Lambda=\begin{bmatrix}0&0&\dots&0&1\\0&0&\dots&1&1\end{bmatrix},$$

which is also a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B=3$.

Now, we need an algorithm to invert the process and calculate $\{\Lambda_i\}_{i\in I}$ from the sequence $\{\Gamma_j^\Lambda\}_{j\in I}$.

Theorem 2.3. *Let $\Xi=\{\Xi_i\}_{i\in I}$ and $\Psi=\{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. Let $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then, for all $i\in I$,*

$$\Lambda_i=\sum_{j\in I}\Psi_i(\Gamma_j^\Lambda)^*\Xi_j.$$

In particular, this shows that $\{\Lambda_i\}_{i\in I}$ is the g-R-dual sequence for $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ψ,Ξ).

Proof. The definition of $\{\Gamma_j^\Lambda\}_{j\in I}$ implies that for every $i,j\in I$

$$\Psi_i(\Gamma_j^\Lambda)^*=\Psi_i\Big(\sum_{k\in I}\Xi_j\Lambda_k^*\Psi_k\Big)^*=\sum_{k\in I}\Psi_i\Psi_k^*\Lambda_k\Xi_j^*=\sum_{k\in I}\delta_{ik}\Lambda_k\Xi_j^*=\Lambda_i\Xi_j^*.$$

Therefore $\Psi_i(\Gamma_j^\Lambda)^*=\Lambda_i\Xi_j^*$. Now, by Lemma 1.5 we have

$$\Lambda_i=\Lambda_i I_{\mathcal{H}}=\Lambda_i\Big(\sum_{j\in I}\Xi_j^*\Xi_j\Big)=\sum_{j\in I}\Lambda_i\Xi_j^*\Xi_j=\sum_{j\in I}\Psi_i(\Gamma_j^\Lambda)^*\Xi_j.$$

□

Definition 2.4. Let $\Xi=\{\Xi_j\}_{j\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and let $\Lambda=\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with the g-frame operator $S_\Lambda:\mathcal{H}\to\mathcal{H}$, respectively. Then the matrix representation of S_Λ with respect to Ξ is the matrix $[S_\Lambda]=[S_{ij}]$, with $S_{ij}=\Xi_i S_\Lambda\Xi_j^*$. Therefore

$$[S_\Lambda]:\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}\to\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2},\quad\text{with}\quad [S_\Lambda f]_\Xi=[S_\Lambda][f]_\Xi,\quad\forall f\in\mathcal{H}.$$

Suppose $A=[A_{ij}]$ with $A_{ij}=\Lambda_i\Xi_j^*$, then $A^*=[A_{ij}^*]$ and $A_{ij}^*=\Xi_i\Lambda_j^*$ for all $i,j\in I$. Therefore

$$A:\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}\to\Big(\sum_{i\in I}\oplus V_i\Big)_{\ell^2},\quad\text{and}\quad A^*A:\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}\to\Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}.$$

The matrix A is called the analysis matrix for Λ with respect to Ξ. A direct calculation shows that for every $f\in\mathcal{H}$ we have $A[f]_\Xi=T_\Lambda f$. We also have

$$[A^*A]_{ij}=\sum_{k\in I}[A^*]_{ik}[A]_{kj}=\sum_{k\in I}\Xi_i\Lambda_k^*\Lambda_k\Xi_j^*=\Xi_i\Big(\sum_{k\in I}\Lambda_k^*\Lambda_k\Big)\Xi_j^*=\Xi_i S_\Lambda\Xi_j^*=S_{ij}=[S_\Lambda]_{ij}.$$

Thus, $A^*A=S_\Lambda$, where $A^*A=S_\Lambda$ means that $A^*A=[S_\Lambda]$.

The following result is a generalization of [4, Proposition 3] to g-frames about dependence of the g-R-dual sequence $\{\Gamma_j^{\Lambda}\}_{j\in J}$ to choose the g-orthonormal bases $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$.

Theorem 2.5. *Let $\Xi = \{\Xi_j\}_{j\in I}$, $\Xi' = \{\Xi'_j\}_{j\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$, $\Psi' = \{\psi'_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$ and let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Denote the analysis matrix for Λ with respect to Ξ by A and the g-R-dual sequences of Λ with respect to (Ξ, Ψ) and (Ξ', Ψ') by $\{\Gamma_j^{\Lambda}\}_{j\in J}, \{\Gamma_j'^{\Lambda}\}_{j\in J}$, respectively. Then the following conditions are equivalent.*

(i) $\Gamma_j^{\Lambda} = \Gamma_j'^{\Lambda}$ *for all* $j \in I$.

(ii) *If* B *and* C *are the transition matrices from* Ξ *to* Ξ' *and* Ψ *to* Ψ'*, respectively, then* $AB^* = CA$.

Proof. Let $B = [B_{ij}]$ and $C = [C_{ij}]$. By the definition of $\{\Gamma_j^{\Lambda}\}_{j\in J}, \{\Gamma_j'^{\Lambda}\}_{j\in J}$ for every $i, j \in I$ we have $\Psi_i(\Gamma_j^{\Lambda})^* = \Lambda_i \Xi_j^*$ and $\Psi'_i(\Gamma_j'^{\Lambda})^* = \Lambda_i \Xi_j'^*$. Since

$$[AB^*]_{ij} = \sum_{k\in I} A_{ik} B^*_{kj} = \sum_{k\in I} \Lambda_i \Xi_k^* \Xi_k \Xi_j'^* = \Lambda_i \Big(\sum_{k\in I} \Xi_k^* \Xi_k \Big) \Xi_j'^* = \Lambda_i \Xi_j'^* = \Psi'_i (\Gamma_j'^{\Lambda})^*$$

and

$$[CA]_{ij} = \sum_{k\in I} C_{ik} A_{kj} = \sum_{k\in I} \Psi'_i \Psi_k^* \Lambda_k \Xi_j^* = \sum_{k\in I} \Psi'_i \Psi_k^* \Psi_k (\Gamma_j^{\Lambda})^* = \Psi'_i \Big(\sum_{k\in I} \Psi_k^* \Psi_k \Big) (\Gamma_j^{\Lambda})^* = \Psi'_i (\Gamma_j^{\Lambda})^*,$$

the conclusion follows. □

Corollary 2.6. *In addition to the hypothesis of Theorem 2.5, if $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ and $\{\Gamma_j^{\Lambda}\}_{j\in I} = \{\Gamma_j'^{\Lambda}\}_{j\in I}$, then $A^*C^*AS_\Lambda^{-1}B^* = I$, where I is the identity matrix.*

Proof. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Definition 2.4 implies that $S_\Lambda^{-1}A^*A = I$. Thus, if $\Gamma_j^{\Lambda} = \Gamma_j'^{\Lambda}$ for all $j \in I$, then by Theorem 2.5, $AB^* = CA$. This implies $B^* = S_\Lambda^{-1}A^*CA$. But B has to be unitary, which yields $A^*C^*AS_\Lambda^{-1}B^* = I$. □

Recall that two sequences $\{\Gamma_j\}_{j\in I}$ and $\{\Gamma'_j\}_{j\in I}$ are called equivalent (unitarily equivalent) in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, if there exists a bounded linear invertible (unitary) operator $T : \mathcal{H} \to \mathcal{H}$ such that $T\Gamma_j^* = \Gamma_j'^*$ for all $j \in I$.

To have a better understanding of the different types of equivalency, we prove the following characterization result.

Theorem 2.7. *In addition to the hypothesis of Theorem 2.5, if $\Gamma = \{\Gamma_j^{\Lambda}\}_{j\in I}$ and $\Gamma' = \{\Gamma_j'^{\Lambda}\}_{j\in I}$ are g-frames for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_j\}_{j\in I}$, respectively, then the following statements hold.*

(i) *If $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, then $\{\Gamma_j^{\Lambda}\}_{j\in I}$ is equivalent to $\{\Gamma_j'^{\Lambda}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if and only if* $\ker(A) = \ker(AB^*)$.

(ii) *$\{\Gamma_j^{\Lambda}\}_{j\in I}$ is unitarily equivalent to $\{\Gamma_j'^{\Lambda}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, if and only if*

$$A^*A = (AB^*)^*(AB^*).$$

Moreover, if $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, then the above is equivalent to $S_\Lambda = BS_\Lambda B^*$.

Proof.

(i) First we observe that, for every $g' = \{g'_k\}_{k\in I} \in \big(\sum_{j\in I} \oplus V_j\big)_{\ell^2}$ we have

$$\sum_{k\in I} \|g'_k\|^2 = \sum_{k\in I} \langle g'_k, g'_k \rangle = \sum_{k\in I} \Big\langle \sum_{i\in I} \Psi'_k \Psi'^*_i g'_i, g'_k \Big\rangle = \Big\langle \sum_{i\in I} \Psi'^*_i g'_i, \sum_{k\in I} \Psi'^*_k g'_k \Big\rangle = \Big\| \sum_{k\in I} \Psi'^*_k g'_k \Big\|^2.$$

Therefore,

$$\sum_{k\in I}\Psi'^{*}_{k}g'_{k}=0\Leftrightarrow g'=0.$$

(Necessity). Suppose that $\{\Gamma_j^{\Lambda}\}_{j\in I}$ is equivalent to $\{\Gamma_j'^{\Lambda}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, then there exists a bounded linear invertible operator $T:\mathcal{H}\to\mathcal{H}$ such that

$$T\big(\sum_{j\in I}(\Gamma_j^{\Lambda})^{*}g_j\big)=\sum_{j\in I}(\Gamma'^{\Lambda}_{j})^{*}g_j,\quad \forall\,\{g_j\}_{j\in I}\in\big(\sum_{j\in I}\oplus W_j\big)_{\ell^2}.$$

Now, $Ag=0$ with $g=\{g_j\}_{j\in I}$, if and only if

$$T^{-1}\big(\sum_{j\in I}(\Gamma'^{\Lambda}_{j})^{*}g_j\big)=\sum_{j\in I}(\Gamma_j^{\Lambda})^{*}g_j=\sum_{k\in I}\sum_{j\in I}\Psi_k^{*}\Lambda_k\Xi_j^{*}g_j=\sum_{k\in I}\sum_{j\in I}\Psi_k^{*}A_{kj}g_j=\sum_{k\in I}\Psi_k^{*}(Ag)_k=0,$$

if and only if

$$\begin{aligned}
\sum_{k\in I}\Psi'^{*}_{k}(AB^{*}g)_k&=\sum_{k\in I}\Psi'^{*}_{k}\big(\sum_{j\in I}[AB^{*}]_{kj}g_j\big)\\
&=\sum_{k\in I}\sum_{j\in I}\sum_{i\in I}\Psi'^{*}_{k}A_{ki}B^{*}_{ij}g_j\\
&=\sum_{k\in I}\sum_{j\in I}\sum_{i\in I}\Psi'^{*}_{k}\Lambda_k\Xi_i^{*}\Xi_i\Xi'^{*}_{j}g_j\\
&=\sum_{k\in I}\sum_{j\in I}\Psi'^{*}_{k}\Lambda_k\big(\sum_{i\in I}\Xi_i^{*}\Xi_i\Xi'^{*}_{j}g_j\big)\\
&=\sum_{k\in I}\sum_{j\in I}\Psi'^{*}_{k}\Lambda_k\Xi'^{*}_{j}g_j=\sum_{j\in I}(\Gamma_j'^{\Lambda})^{*}g_j=TT^{-1}\big(\sum_{j\in I}(\Gamma_{j'}^{\Lambda})^{*}g_j\big)=0,
\end{aligned}$$

if and only if $AB^{*}g=0$.

(Sufficiency). Suppose that $\ker(A)=\ker(AB^{*})$. Define the operator T as follows:

$$T:\operatorname{span}\big\{(\Gamma_j^{\Lambda})^{*}(W_j)\big\}_{j\in I}\to\operatorname{span}\big\{(\Gamma'^{\Lambda}_{j})^{*}(W_j)\big\}_{j\in I},\quad T\big(\sum_{j\in J}(\Gamma_j^{\Lambda})^{*}g_j\big)=\sum_{j\in J}(\Gamma_j'^{\Lambda})^{*}g_j,$$

for all $J\subset I$ with $|J|<\infty$ and $g_j\in W_j$ $(j\in J)$. Let $C,D>0$ be the g-frame bounds for g-frame $\Lambda=\{\Lambda_i\}_{i\in I}$. Then we have

$$\begin{aligned}
\|T\big(\sum_{j\in J}(\Gamma_j^{\Lambda})^{*}g_j\big)\|^2&=\|\sum_{j\in J}(\Gamma_j'^{\Lambda})^{*}g_j\|^2=\|\sum_{k\in I}\sum_{j\in J}\Psi'^{*}_{k}\Lambda_k\Xi'^{*}_{j}g_j\|^2\\
&=\|\sum_{k\in I}\Psi'^{*}_{k}\Lambda_k\big(\sum_{j\in J}\Xi'^{*}_{j}g_j\big)\|^2=\sum_{k\in I}\|\Lambda_k\big(\sum_{j\in J}\Xi'^{*}_{j}g_j\big)\|^2\\
&\leqslant D\|\sum_{j\in J}\Xi'^{*}_{j}g_j\|^2=D\sum_{j\in J}\|g_j\|^2=D\|\sum_{j\in J}\Xi_j^{*}g_j\|^2\\
&\leqslant\frac{D}{C}\sum_{k\in I}\|\Lambda_k\big(\sum_{j\in J}\Xi_j^{*}g_j\big)\|^2=\frac{D}{C}\|\sum_{k\in I}\Psi_k^{*}\Lambda_k\big(\sum_{j\in J}\Xi_j^{*}g_j\big)\|^2\\
&=\frac{D}{C}\|\sum_{j\in J}\big(\sum_{k\in I}\Xi_j\Lambda_k^{*}\Psi_k\big)^{*}g_j\|^2=\frac{D}{C}\|\sum_{j\in J}(\Gamma_j^{\Lambda})^{*}g_j\|^2.
\end{aligned}$$

This shows that T is a bounded linear operator. To prove invertibility of T we compute

$$\begin{aligned}
T\big(\sum_{j\in J}(\Gamma_j^{\Lambda})^{*}g_j\big)&=\sum_{j\in J}(\Gamma_j'^{\Lambda})^{*}g_j=\sum_{k\in I}\sum_{j\in J}\Psi'^{*}_{k}\Lambda_k\Xi'^{*}_{j}g_j=\sum_{k\in I}\sum_{j\in J}\Psi'^{*}_{k}\Lambda_k\big(\sum_{i\in I}\Xi_i^{*}\Xi_i\Xi'^{*}_{j}g_j\big)\\
&=\sum_{k\in I}\Psi'^{*}_{k}\big(\sum_{j\in J}[AB^{*}]_{kj}g_j\big)=\sum_{k\in I}\Psi'^{*}_{k}(AB^{*}g)_k.
\end{aligned}$$

We also have

$$\sum_{j\in J}(\Gamma_j^{\Lambda})^* g_j = \sum_{k\in I}\sum_{j\in J}\Psi_k^*\Lambda_k\Xi_j^* g_j = \sum_{k\in I}\Psi_k^*(Ag)_k.$$

Hence,

$$T\Big(\sum_{j\in J}(\Gamma_j^{\Lambda})^* g_j\Big) = 0 \Leftrightarrow \sum_{j\in J}(\Gamma_j^{\Lambda})^* g_j = 0.$$

This implies that T is invertible operator. Now, the g-completeness of Γ and Γ' for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ implies that T has an extension invertible on $\mathcal{H}$ and $T(\Gamma_j^{\Lambda})^* = ({\Gamma'}_j^{\Lambda})^*$ for all $j \in I$.

(ii) First, we prove $[A^*A]_{ij} = \Gamma_i^{\Lambda}(\Gamma_j^{\Lambda})^*$ and $[(AB^*)^*(AB^*)]_{ij} = {\Gamma'}_i^{\Lambda}({\Gamma'}_j^{\Lambda})^*$. To see this, we have

$$\begin{aligned}\Gamma_i^{\Lambda}(\Gamma_j^{\Lambda})^* &= \Big(\sum_{k\in I}\Xi_i\Lambda_k^*\Psi_k\Big)\Big(\sum_{m\in I}\Psi_m^*\Lambda_m\Xi_j^*\Big) = \sum_{k\in I}\sum_{m\in I}\delta_{km}\Xi_i\Lambda_k^*\Lambda_m\Xi_j^* = \sum_{k\in I}\Xi_i\Lambda_k^*\Lambda_k\Xi_j^*\\ &= \sum_{k\in I}A_{ik}^*A_{kj} = [A^*A]_{ij}.\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}{\Gamma'}_i^{\Lambda}({\Gamma'}_j^{\Lambda})^* &= \Big(\sum_{k\in I}\Xi_i'\Lambda_k^*\Psi_k'\Big)\Big(\sum_{m\in I}\Psi_m'^*\Lambda_m\Xi_j'^*\Big)\\ &= \sum_{k\in I}\sum_{m\in I}\delta_{km}{\Xi'}_i\Lambda_k^*\Lambda_m\Xi_j'^* = \sum_{k\in I}(\Lambda_k\Xi_i'^*)^*(\Lambda_k{\Xi'}_j^*)\\ &= \sum_{k\in I}\Big(\sum_{n\in I}\Lambda_k\Xi_n^*\Xi_n{\Xi'}_i^*\Big)^*\Big(\sum_{m\in I}\Lambda_k\Xi_m^*\Xi_m{\Xi'}_j^*\Big)\\ &= \sum_{k\in I}\Big(\sum_{n\in I}A_{kn}B_{ni}^*\Big)^*\Big(\sum_{m\in I}A_{km}B_{mj}^*\Big)\\ &= \sum_{k\in I}(AB^*)_{ik}^*(AB^*)_{kj} = [(AB^*)^*(AB^*)]_{ij}.\end{aligned}$$

Now, let $A^*A = (AB^*)^*(AB^*)$. Define the operator T as follows:

$$T : \text{span}\big\{(\Gamma_j^{\Lambda})^*(W_j)\big\}_{j\in I} \to \text{span}\big\{({\Gamma'}_j^{\Lambda})^*(W_j)\big\}_{j\in I}, \quad T\Big(\sum_{j\in J}(\Gamma_j^{\Lambda})^* g_j\Big) = \sum_{j\in J}({\Gamma'}_j^{\Lambda})^* g_j,$$

for all finite subsets $J \subset I$ and $g_j \in W_j$ $(j \in J)$. Let $f_1, f_2 \in \text{span}\big\{(\Gamma_j^{\Lambda})^*(W_j)\big\}_{j\in I}$ as $f_1 = \sum_{j\in J_1}(\Gamma_j^{\Lambda})^* g_{1j}$ and $f_2 = \sum_{j\in J_2}(\Gamma_j^{\Lambda})^* g_{2j}$, we have

$$\begin{aligned}\langle Tf_1, Tf_2\rangle &= \Big\langle \sum_{j\in J_1}({\Gamma'}_j^{\Lambda})^* g_{1j}, \sum_{k\in J_2}({\Gamma'}_k^{\Lambda})^* g_{2k}\Big\rangle\\ &= \sum_{j\in J_1}\sum_{k\in J_2}\langle {\Gamma'}_k^{\Lambda}({\Gamma'}_j^{\Lambda})^* g_{1j}, g_{2k}\rangle\\ &= \Big\langle \sum_{j\in J_1}(\Gamma_j^{\Lambda})^* g_{1j}, \sum_{k\in J_2}(\Gamma_k^{\Lambda})^* g_{2k}\Big\rangle\\ &= \langle f_1, f_2\rangle.\end{aligned}$$

This implies that T is a bounded linear surjective isometry operator. Thus, the g-completeness of Γ and Γ' for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ implies that T has an extension isometry on $\mathcal{H}$ and $T(\Gamma_j^{\Lambda})^* = ({\Gamma'}_j^{\Lambda})^*$ for all $j \in I$. This shows that Γ is unitarily equivalent to Γ' in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. The converse implication is obvious. Finally, if $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, then, since $A^*A = S_\Lambda$, thus

$$S_\Lambda = A^*A = (AB^*)^*(AB^*) = BA^*AB^* = BS_\Lambda B^*.$$

□

3. Characterizations of equivalence of the g-R-dual sequence

In this section we first characterize all sequences with lower g-frame bound. Next, we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

Recall that a family $\{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$, if it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(V_i)\}_{i\in I}$ with respect to $\{V_i\}_{i\in I}$.

There exists a characterization of frames which keeps the information about the frame bounds ([5, Lemma 5.5.5]). A similar result holds in g-frame situation.

Proposition 3.1. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\}$. Then the following conditions are equivalent.*

(i) *$\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A and B.*
(ii) *The synthesis operator T_Λ^* is well-defined on $\big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ such that:*

$$A\|g'\|_{\ell^2}^2 \leqslant \|T_\Lambda^* g'\|^2 \leqslant B\|g'\|_{\ell^2}^2, \quad \forall\, g' \in (\ker_{T_\Lambda^*})^\perp.$$

Proof. This follows immediately from [5, Lemma 5.5.5]. □

The next result shows a basic connection between a sequence of operators and its g-R-dual sequence which will be used frequently in what follows.

Theorem 3.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then for every $\{g_j\}_{j\in I} \in \big(\sum_{j\in I}\oplus W_j\big)_{\ell^2}$, $\{g_i'\}_{i\in I} \in \big(\sum_{i\in I}\oplus V_i\big)_{\ell^2}$ satisfying $f = \sum_{j\in I}\Xi_j^* g_j$ and $h = \sum_{i\in I}\Psi_i^* g_i'$, we have*

$$\Big\|\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j\Big\|^2 = \sum_{i\in I}\|\Lambda_i f\|^2 \quad \textit{and} \quad \Big\|\sum_{i\in I}\Lambda_i^* g_i'\Big\|^2 = \sum_{j\in I}\|\Gamma_j^\Lambda h\|^2.$$

Proof. It is easy to check that

$$\begin{aligned}\Big\|\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j\Big\|^2 &= \Big\|\sum_{j\in I}\Big(\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i\Big)^* g_j\Big\|^2 = \Big\|\sum_{i\in I}\Psi_i^*\Lambda_i f\Big\|^2 = \Big\langle\sum_{i\in I}\Psi_i^*\Lambda_i f, \sum_{j\in I}\Psi_j^*\Lambda_j f\Big\rangle \\ &= \sum_{i\in I}\sum_{j\in I}\langle\Lambda_i f, \Psi_i\Psi_j^*\Lambda_j f\rangle \\ &= \sum_{i\in I}\sum_{j\in I}\langle\Lambda_i f, \delta_{ij}\Lambda_j f\rangle = \sum_{i\in I}\|\Lambda_i f\|^2.\end{aligned}$$

Similarly, the second claim follows from Theorem 2.3. □

Corollary 3.3. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

$$\|T_{\Gamma^\Lambda}^*\big([f]_\Xi\big)\| = \|T_\Lambda f\|_{\ell^2}, \quad \|T_\Lambda^*\big([f]_\Psi\big)\| = \|T_{\Gamma^\Lambda} f\|_{\ell^2},$$

for every $f \in \mathcal{H}$.

Proof. This follows immediately from Theorem 3.2. □

There exists an interesting relation between the synthesis operator of $\Lambda = \{\Lambda_i\}_{i\in I}$ and the span of $\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}$, which will turn out to be very useful in the sequel.

Theorem 3.4. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-R-dual sequence $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ξ, Ψ). Then the following statements hold.*

(i) $f \in \big(\overline{\text{span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}\big)^\perp$ *if and only if* $[f]_\Psi \in \ker T_\Lambda^*$.

(ii) $f \in \left(\overline{\text{span}}\{\Lambda_j^*(V_j)\}_{j\in I}\right)^{\perp}$ *if and only if* $[f]_{\Xi} \in \ker T_{\Gamma^{\Lambda}}^*$.

Proof. Let $f \in \mathcal{H}$. First for each $j \in J$ and $g_j \in W_j$ we observe that

$$\langle f, (\Gamma_j^{\Lambda})^* g_j\rangle = \sum_{i\in J}\langle f, \Psi_i^*\Lambda_i\Xi_j^* g_j\rangle = \Big\langle \sum_{i\in J}\Lambda_i^*\Psi_i f, \Xi_j^* g_j\Big\rangle = \big\langle T_{\Lambda}^*([f]_{\Psi}), \Xi_j^* g_j\big\rangle.$$

Since $\Xi = \{\Xi_j\}_{j\in J}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, $\big\langle T_{\Lambda}^*([f]_{\Psi}), \Xi_j^* g_j\big\rangle = 0$ for all $j \in I$ and $g_j \in W_j$, if and only if $T_{\Lambda}^*([f]_{\Psi}) = 0$. Thus, $f \in \left(\text{span}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp}$ is equivalent to $[f]_{\Psi} \in \ker T_{\Lambda}^*$. Similarly, the second claim follows from Theorem 2.3. □

Corollary 3.5. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-R-dual sequence $\{\Gamma_j^{\Lambda}\}_{j\in I}$ with respect to (Ξ, Ψ). Then*

$$\dim\left(\overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp} = \dim\ker T_{\Lambda}^* \quad \textit{and} \quad \dim\left(\overline{\text{span}}\{\Lambda_j^*(V_j)\}_{j\in I}\right)^{\perp} = \dim\ker T_{\Gamma^{\Lambda}}^*.$$

Proof. This follows immediately from Theorem 3.4. □

The next result shows a kind of equilibrium between a sequence of operators and its R-dual sequence. It can be viewed as a general version of [4, Proposition 13].

Corollary 3.6. *The following conditions are equivalent.*

(i) $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a g-frame sequence with respect to* $\{V_i\}_{i\in I}$ *with g-frame bounds* A, B.
(ii) $\{\Gamma_j^{\Lambda}\}_{j\in I}$ *is a g-frame sequence with respect to* $\{W_j\}_{j\in I}$ *with g-frame bounds* A, B.
(iii) $\{\Gamma_j^{\Lambda}\}_{j\in I}$ *is a g-Riesz basic sequence with respect to* $\{W_j\}_{j\in I}$ *with g-frame bounds* A, B.

Proof. (i)⇔ (ii). The Proposition 3.1 and Theorem 3.4 conclude that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A, B if and only if

$$A\|[f]_{\Psi}\|_{\ell^2}^2 \leqslant \|T_{\Lambda}^*([f]_{\Psi})\|^2 \leqslant B\|[f]_{\Psi}\|_{\ell^2}^2,$$

for all $f \in \overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}$. Now, Corollary 3.3 implies

$$A\|f\|^2 \leqslant \|T_{\Gamma^{\Lambda}} f\|_{\ell^2}^2 \leqslant B\|f\|^2.$$

(i)⇔(iii). This equivalence follows immediately from Theorem 3.2. □

The dimension condition in Corollary 3.5 will play a crucial role for the g-R-dual sequence. Using Corollary 3.5 we can derive a simple characterization of a g-Riesz basic sequence being a g-R-dual sequence of a g-frame in the tight case.

Theorem 3.7. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a A-tight g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ and let $\{\Gamma_j\}_{j\in I}$ be an A-tight g-Riesz basic sequence in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Then $\{\Gamma_j\}_{j\in I}$ is a g-R-dual sequence of $\{\Lambda_i\}_{i\in I}$ with respect to (Ξ, Ψ), if and only if*

$$\dim\left(\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^{\perp} = \dim\ker T_{\Lambda}^*. \tag{3.1}$$

Proof. The necessity of the condition in (3.1) follows from Corollary 3.5. Now, assume that (3.1) holds. Then, according to Lemma 1.6 the sequence $\{\frac{1}{\sqrt{A}}\Gamma_j\}_{j\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Suppose that $\Xi = \{\Xi_j\}_{j\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ are g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$, respectively. Consider the g-R-dual $\{\Theta_j\}_{j\in I}$ of $\Lambda = \{\Lambda_i\}_{i\in I}$ with respect to (Ξ, Ψ), i.e., $\Theta_j = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i$, $j \in I$. By Corollary 3.6 $\{\Theta_j\}_{j\in I}$ is an A-tight g-Riesz basic sequence with respect to

$\{W_j\}_{j\in I}$ and hence $\{\frac{1}{\sqrt{A}}\Theta_j\}_{j\in I}$ is also a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. By Corollary 3.5 and (3.1),

$$\dim\left(\overline{\operatorname{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^{\perp} = \dim\ker T_\Lambda^* = \dim\left(\overline{\operatorname{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^{\perp}. \tag{3.2}$$

In case $\left(\overline{\operatorname{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^{\perp} = \left(\overline{\operatorname{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^{\perp} = \{0\}$, the g-orthonormality of the sequences $\{\frac{1}{\sqrt{A}}\Theta_i\}_{i\in I}$ and $\{\frac{1}{\sqrt{A}}\Gamma_i\}_{i\in I}$ implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \ \ \forall j \in I.$$

In case $\left(\overline{\operatorname{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^{\perp} \neq \{0\}$, letting $\{\Phi_j\}_{j\in I}$ and $\{\Omega_j\}_{j\in I}$ be g-orthonormal bases for

$$\left(\overline{\operatorname{span}}\{\Theta_j^*(W_j)\}_{j\in I}\right)^{\perp} \quad \text{and} \quad \left(\overline{\operatorname{span}}\{\Gamma_j^*(W_j)\}_{j\in I}\right)^{\perp},$$

with respect to $\{W_j\}_{j\in I}$, respectively, (3.2) implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \quad \Omega_j = \Phi_j U^* \ \ \forall j \in I.$$

In both cases, we have

$$\Gamma_j = \Theta_j U^* = \Big(\sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i\Big) U^* = \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i U^*, \ \ \forall j \in I,$$

which shows that $\{\Gamma_j\}_{j\in I}$ is a g-R-dual sequence of $\{\Lambda_i\}_{i\in I}$ with respect to $\{\Xi_j\}_{j\in I}$ and $\{\Psi_i U^*\}_{i\in I}$. □

The following result is about different types of equivalence of g-frames, which is taken from [12]. This result will moreover be employed in several proofs in the sequel.

Proposition 3.8. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Lambda' = \{\Lambda_i'\}_{i\in I}$ be Parseval g-frames for $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to $\{V_i\}_{i\in I}$, respectively. Then Λ is unitarily equivalent to Λ' if and only if the analysis operators T_Λ and $T_{\Lambda'}$ have the same range. Likewise, two g-frames with respect to $\{V_i\}_{i\in I}$ are equivalent if and only if their analysis operators have the same range.*

In the following we characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

Theorem 3.9. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda_i'\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

(i) *$\{\Lambda_i\}_{i\in I}$ is equivalent to $\{\Lambda_i'\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if*

$$\overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I} = \overline{\operatorname{span}}\{(\Gamma_j^{\Lambda'})^*(W_j)\}_{j\in I};$$

(ii) *$\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda_i'\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$;*

(iii) *$\{\Gamma_j^{\Lambda}\}_{j\in I}$ is unitarily equivalent to $\{\Gamma_j^{\Lambda'}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if and only if $S_\Lambda = S_{\Lambda'}$.*

Proof.

(i) By Proposition 3.8, $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda_i'\}_{i\in I}$ are equivalent in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, if and only if $\mathcal{R}_{T_\Lambda} = \mathcal{R}_{T_{\Lambda'}}$ and hence $\ker T_\Lambda^* = \ker T_{\Lambda'}^*$. Now the claim follows from Theorem 3.4.

(ii) Using Propositions 3.1 and 3.8, $\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda_i'\}_{i\in I}$ if and only if

$$\Big\|\sum_{i\in I} \Lambda_i^* g_i'\Big\|^2 = \Big\|\sum_{i\in I} \Lambda'^*_i g_i'\Big\|^2, \quad \forall \{g_i'\}_{i\in I} \in (\ker T_\Lambda^*)^{\perp}.$$

By Theorem 3.2, this in turn is equivalent to

$$\langle S_{\Gamma^\Lambda} f, f\rangle = \sum_{j\in I} \|\Gamma_j^{\Lambda} f\|^2 = \sum_{j\in I} \|\Gamma_j^{\Lambda'} f\|^2 = \langle S_{\Gamma^{\Lambda'}} f, f\rangle,$$

for all $f \in \mathcal{H}$ and $g_i' = \Psi_i f$ $(i \in I)$. It follows that $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$, as required.

(iii) The proof follows immediately from (ii) and Theorem 2.3. □

Corollary 3.10. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

$$\overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I} = \overline{\text{span}}\{(\Gamma_j^{\widehat{\Lambda}})^*(W_j)\}_{j\in I},$$

where $\{\widehat{\Lambda}_i\}_{i\in I}$ is the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Proof. Since $\{\widehat{\Lambda}_i\}_{i\in I}$ is equivalent to $\{\Lambda_i\}_{i\in I}$, this claim follows from Theorem 3.9. □

4. Duality properties of the g-R-dual sequence

In this section we characterize all properties of a g-Bessel sequence in terms of properties of their g-R-dual sequence. We will study properties of dual g-frames and canonical dual g-frames. This is a general version of duality principle for g-frames which follows from the Casazza duality relations [4].

The next result gives an explicit form for g-R-dual sequence of the canonical dual g-frame.

Theorem 4.1. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Omega_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if g-R-dual sequences $\{\Gamma_j^{\Lambda}\}_{j\in I}$ and $\{\Gamma_j^{\Omega}\}_{j\in I}$ are g-biorthogonal, i.e.,*

$$\Gamma_i^{\Lambda}(\Gamma_j^{\Omega})^* g_j = \Gamma_i^{\Omega}(\Gamma_j^{\Lambda})^* g_j = \delta_{ij} g_j, \quad \forall\, i,j \in I,\ g_j \in W_j.$$

Proof. Let $\{\Omega_i\}_{i\in I}$ be a dual g-frame of $\{\Lambda_i\}_{i\in I}$. By definition of $\{\Gamma_j^{\Omega}\}_{j\in I}$ and $\{\Gamma_j^{\Lambda}\}_{j\in I}$ for every $i,j \in I$ and $g_j \in W_j$ we have

$$\begin{aligned}\Gamma_i^{\Lambda}(\Gamma_j^{\Omega})^* g_j &= \sum_{k\in I} \Xi_i \Lambda_k^* \Psi_k \Big(\sum_{m\in I} \Xi_j \Omega_m^* \Psi_m\Big)^* g_j \\ &= \sum_{k\in I}\sum_{m\in I} \Xi_i \Lambda_k^* \Psi_k \Psi_m^* \Omega_m \Xi_j^* g_j \\ &= \sum_{k\in I} \Xi_i \Lambda_k^* \Omega_k \Xi_j^* g_j = \Xi_i \Big(\sum_{k\in I} \Lambda_k^* \Omega_k \Xi_j^* g_j\Big) = \Xi_i \Xi_j^* g_j = \delta_{ij} g_j.\end{aligned}$$

The converse implication similarly follows from Theorem 2.3. □

Corollary 4.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with canonical dual g-frame denoted by $\{\widehat{\Lambda}_i\}_{i\in I}$. Then the g-R-dual sequences $\{\Gamma_j^{\Lambda}\}_{j\in I}$ and $\{\Gamma_j^{\widehat{\Lambda}}\}_{j\in I}$ are g-biorthogonal, i.e.,*

$$\Gamma_i^{\Lambda}(\Gamma_j^{\widehat{\Lambda}})^* g_j = \Gamma_i^{\widehat{\Lambda}}(\Gamma_j^{\Lambda})^* g_j = \delta_{ij} g_j$$

for all $i,j \in I$ and $g_j \in W_j$. Thus $\{\Gamma_j^{\widehat{\Lambda}}\}_{j\in I}$ is the dual g-Riesz basic sequence of $\{\Gamma_j^{\Lambda}\}_{j\in I}$.

The next result is a characterization of tight g-frames in terms of their g-R-dual sequence.

Corollary 4.3. *$\{\Lambda_i\}_{i\in I}$ is an A-tight g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if g-R-dual sequence $\{\frac{1}{\sqrt{A}}\Gamma_j^{\Lambda}\}_{j\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Thus the sequence $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame if and only if, its g-R-dual sequence is an orthonormal system.*

Proof. This follows immediately from Lemma 1.6, Corollary 3.6, and Theorem 4.2. □

Theorem 4.4. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Omega_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if, there exists a g-Bessel sequence $\{\Theta_j\}_{j\in I}$ for $\big(\overline{\text{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\big)^{\perp}$ with respect to $\{W_j\}_{j\in I}$, such that $\Gamma_j^{\Omega} = \Gamma_j^{\widehat{\Lambda}} + \Theta_j$ for all $j \in I$.*

Proof. Suppose that $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$. By Theorem 4.1 we have

$$\begin{aligned}\left\langle (\Gamma_i^{\Omega}-\Gamma_i^{\widehat{\Lambda}})^* g_i, (\Gamma_j^{\Lambda})^* g_j \right\rangle &= \left\langle g_i, (\Gamma_i^{\Omega}-\Gamma_i^{\widehat{\Lambda}})(\Gamma_j^{\Lambda})^* g_j \right\rangle = \left\langle g_i, \Gamma_i^{\Omega}(\Gamma_j^{\Lambda})^* g_j \right\rangle - \left\langle g_i, \Gamma_i^{\widehat{\Lambda}}(\Gamma_j^{\Lambda})^* g_j \right\rangle \\ &= \langle g_i, \delta_{ij} g_j \rangle - \langle g_i, \delta_{ij} g_j \rangle = 0,\end{aligned}$$

for all $i, j \in I$ and $g_i \in W_i, g_j \in W_j$. Thus, Definition 2.1 implies that $\Theta_j = \Gamma_j^{\Omega} - \Gamma_j^{\widehat{\Lambda}}$ is a g-Bessel sequence for $\left(\overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp}$ with respect to $\{W_j\}_{j\in I}$ and $\Gamma_j^{\Omega} = \Gamma_j^{\widehat{\Lambda}} + \Theta_j$. Now for the opposite implication, suppose that there exists a g-Bessel sequence $\{\Theta_j\}_{j\in I}$ for $\left(\overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp}$ with respect to $\{W_j\}_{j\in I}$, such that $\Gamma_j^{\Omega} = \Gamma_j^{\widehat{\Lambda}} + \Theta_j$ for all $j \in I$. By Theorem 2.3, we have

$$\Omega_i = \widehat{\Lambda}_i + \sum_{j\in I} \Psi_i(\Theta_j)^* \Xi_j \quad \text{for all } i \in I.$$

So, for each $f \in \mathcal{H}$

$$\sum_{i\in I} \Lambda_i^* \Omega_i f = \sum_{i\in I} \Lambda_i^* \big(\widehat{\Lambda}_i + \sum_{j\in I} \Psi_i \Theta_j^* \Xi_j\big) f = \sum_{i\in I} \Lambda_i^* \widehat{\Lambda}_i f + \sum_{i\in I}\sum_{j\in I} \Lambda_i^* \Psi_i \Theta_j^* \Xi_j f = f + \sum_{j\in I}\sum_{i\in I} \Lambda_i^* \Psi_i \Theta_j^* \Xi_j f,$$

since $\Theta_j^* \Xi_j f \in \left(\overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp}$ for all $j \in I$. Theorem 3.4 implies that

$$\sum_{i\in I} \Lambda_i^* \Psi_i \Theta_j^* \Xi_j f = 0.$$

This proves that $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$. □

Among the dual g-frames the canonical dual g-frame is distinguished by the following properties.

Theorem 4.5. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with canonical dual g-frame denoted by $\{\widehat{\Lambda}_i\}_{i\in I}$ and let $\{\Omega_i\}_{i\in I}$ be a dual g-frame of $\{\Lambda_i\}_{i\in I}$. Then*

$$\|\Gamma_j^{\widehat{\Lambda}}\| \leqslant \|\Gamma_j^{\Omega}\| \quad \textit{for all } j \in I,$$

with equality if and only if $\{\Omega_j\}_{j\in I} = \{\widehat{\Lambda}_j\}_{j\in I}$.

Proof. By Theorem 4.4, $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if $\Gamma_j^{\Omega} = \Gamma_j^{\widehat{\Lambda}} + \Theta_j$, where $(\Gamma_j^{\widehat{\Lambda}})^* g \in \overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}$ and $\Theta_j^* g \in \left(\overline{\operatorname{span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}\right)^{\perp}$ for all $j \in I, g \in W_j$. Hence

$$\begin{aligned}\|\Gamma_j^{\Omega}\|^2 = \|(\Gamma_j^{\Omega})^*\|^2 = \sup_{\|g\|=1} \|(\Gamma_j^{\Omega})^* g\|^2 &= \sup_{\|g\|=1} \|(\Gamma_j^{\widehat{\Lambda}})^* g\|^2 + \sup_{\|g\|=1} \|\Theta_j^* g\|^2 \\ &= \|(\Gamma_j^{\widehat{\Lambda}})^*\|^2 + \|\Theta_j^*\|^2 = \|\Gamma_j^{\widehat{\Lambda}}\|^2 + \|\Theta_j\|^2 \geqslant \|\Gamma_j^{\widehat{\Lambda}}\|^2,\end{aligned}$$

with equality if and only if $\{\Omega_j\}_{j\in I} = \{\widehat{\Lambda}_j\}_{j\in I}$. □

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