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A minimax inequality for a class of functionals and its applications to existence of multiple solutions for elliptic equations

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Abstract

In this paper, we establish an equivalent statement of minimax inequality for a special class of functionals. As an application, we discuss the existence of three solutions to the Dirichlet problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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1 Introduction

Throughout the sequel, $\Omega \subset R^N (N \geq 1)$ is a nonempty bounded open set with smooth boundary $\partial\Omega$ and $p > N$.

Given two Gâteaux differentiable functionals Φ and Ψ on a real Banach space X, the minimax inequality

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda(\rho - \Psi(u))), \ \rho \in R, \tag{1}$$

plays a fundamental role for establishing the existence of at least three critical points for the functional $\Phi(u) - \lambda\Psi(u)$.

In this work some conditions that ensure the minimax inequality (1) are pointed out and equivalent formulations are proved. Lemma 2.1 establishes an equivalent statement of the minimax inequality (1) for a special class of functionals, while its consequences (Lemma 2.3 and Lemma 2.5) guarantee some conditions so that the minimax inequality (1) holds. The technique used in our proof has been introduced in [4].

Finally, as an application of our results, we study the following Dirichlet boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $f : \Omega \times R \to R$ is an L^1-Carath$\acute{e}$odory function and $a(x) \in C(\overline{\Omega})$ is a positive function. and $\lambda > 0$.

We say that u is a weak solution to the problem (2) if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) dx - \lambda \int_\Omega f(x, u(x)) v(x) dx = - \int_\Omega a(x)|u(x)|^{p-2} u(x) v(x) dx$$

for every $v \in W_0^{1,p}(\Omega)$.

In recent years, many authors have studied multiple solutions from several points of view and with variational methods and critical point theory and we refer to [1, 2, 4, 5, 6]and the references therein.

For basic notations and definitions we refer to [9].

We now recall the three critical points theorem of B. Ricceri [7] by choosing $h(\lambda) = \lambda\rho$:

Theorem 1.1. Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $T : X \longrightarrow R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.
Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda T(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda T(u) + \lambda\rho) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda T(u) + \lambda\rho).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda T'(u) = 0$$

has at least three solutions in X whose norms are less that than q.

2 Main results

In the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ equipped with the norm

$$|| \, u \, || := \Big(\int_\Omega |\nabla u(x)|^p dx \Big)^{1/p},$$

$f : \Omega \times R \to R$ is an L^1-Carathéodory function and $g : \Omega \times R \to R$ is the function defined as follows

$$g(x,t) = \int_0^t f(x,\xi) d\xi$$

for each $(x,t) \in \Omega \times R$.

Now, we define

$$||u||_* := \Big(\int_\Omega (|\nabla u|^p + a(x)|u|^p) dx \Big)^{1/p},$$

so by using the positivity of the function $a(x) \in C(\overline{\Omega})$, there exist positive suitable constants c_1 and c_2 :

$$c_1 ||u|| \leq ||u||_* \leq c_2 ||u|| \tag{3}$$

(i.e., the above norms are equivalent).

We now introduce two positive special functionals on the Sobolev space X as follows

$$\Phi(u) := \frac{||u||_*^p}{p}$$

and

$$\Psi(u) := \int_\Omega g(x, u(x)) dx$$

for every $u \in X$.

Let $\rho, r \in R$, $w \in X$ be such that $0 < \rho < \Psi(w)$ and $0 < r < \Phi(w)$. We put

$$A_1(\rho, w) := \rho \frac{\Phi(w)}{\Psi(w)}, \tag{4}$$

$$A_2(r, w) := r \frac{\Psi(w)}{\Phi(w)} \tag{5}$$

and

$$A_3(\rho, w) := \frac{k}{c_1} (p \; A_1(\rho, w))^{1/p}, \tag{6}$$

where $k = k(N,p)$ is a positive constant. Clearly, $A_1(\rho, w)$, $A_2(r, w)$ and $A_3(\rho, w)$ are positive. Now, we put

$$\delta_1; = \inf\{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+; \Psi(u) \geq \rho\},$$

$$\delta_2; = \inf\{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+;\ m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}||u||_*,\ \frac{k}{c_1}||u||_*]} g(x,t) \geq \rho\}$$

and

$$\delta_\rho; = \delta_1 - \delta_2. \tag{7}$$

We prove $\delta_\rho \geq 0$ as follows:

Taking into account that for every $u \in X$, one has

$$\sup_{x\in\Omega} |u(x)| \leq k||u||$$

for each $u \in X$, namely

$$\sup_{x\in\Omega} |u(x)| \leq \frac{k}{c_1}||u||_*$$

for each $u \in X$, so that

$$\Psi(u) = \int_\Omega g(x, u(x))dx \leq m(\Omega) \sup g(x,t)$$

where $(x,t) \in \Omega \times [-\frac{k}{c_1} \mid\mid u \mid\mid_*,\ \frac{k}{c_1} \mid\mid u \mid\mid_*]$. Namely

$$\Psi(u) \leq m(\Omega) \sup g(x,t),$$

where $(x,t) \in \Omega \times [-\frac{k}{c_1} \mid\mid u \mid\mid_*,\ \frac{k}{c_1} \mid\mid u \mid\mid_*]$; therefore,

$$\{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+; \Psi(u) \geq \rho\} \subseteq \{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+; m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}||u||_*\ ,\ \frac{k}{c_1}||u||_*\ ]} g(x,t) \geq \rho\ \}.$$

So, we have

$\inf\{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+; \Psi(u) \geq \rho\} \geq$

$$\inf\{\frac{k}{c_1} \mid\mid u \mid\mid_* \in R^+; m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}||u||_*\ ,\ \frac{k}{c_1}||u||_*\ ]} g(x,t) \geq \rho\ \}.$$

Hence $\delta_\rho \geq 0$.

Now, our main results fully depend on the following lemma:

Lemma 2.1. Assume that there exist $\rho \in R,\ w \in X$ that

(i) $0 < \rho < \Psi(w)$,
(ii) $m(\Omega)\sup_{(x,t)\in\Omega\times[-A_3(\rho,w)+\delta_\rho\ ,\ A_3(\rho,w)-\delta_\rho]} g(x,t) < \rho$;

where $A_3(\rho,w)$ is given by (6) and δ_ρ by (7).
Then, there exists $\rho \in R$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho-\Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda(\rho-\Psi(u))).$$

Our proof of this lemma is motivated by that of [6, Proposition 2.1].
Proof: From (ii), we obtain

$$A_3(\rho,w)-\delta_\rho \notin \{l\in R^+;\ m(\Omega)\sup_{(x,t)\in\Omega\times[-l,l]} g(x,t)\geq \rho\}.$$

Moreover

$$\inf\{l\in R^+; m(\Omega)\sup_{(x,t)\in\Omega\times[-l,l]} g(x,t)\geq\rho\}\geq A_3(\rho,w)-\delta_\rho;$$

in fact, arguing by contradiction, we assume that there is $l_1\in R^+$ such that

$$m(\Omega)\sup_{(x,t)\in\Omega\times[-l_1,l_1]} g(x,t)\geq\rho$$

and

$$l_1 < A_3(\rho,w)-\delta_\rho,$$

so

$$m(\Omega)\sup_{(x,t)\in\Omega\times[-A_3(\rho,w)+\delta_\rho\ ,\ A_3(\rho,w)-\delta_\rho]} g(x,t)\geq m(\Omega)\sup_{(x,t)\in\Omega\times[-l_1,l_1]} g(x,t)\geq\rho$$

and this is a contradiction. So

$$\inf\{l\in R^+; m(\Omega)\sup_{(x,t)\in\Omega\times[-l,l]} g(x,t)\geq\rho\} > A_3(\rho,w)-\delta_\rho.$$

Therefore,

$$\inf\{\frac{k}{c_1}\parallel u\parallel_*\in R^+; m(\Omega)\sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}||u||_*\ ,\ \frac{k}{c_1}||u||_*]} g(x,t)\geq\rho\} > A_3(\rho,w)-\delta_\rho;$$

namely $A_3(\rho,w)<\delta_1$. So, we have

$$\inf\{\frac{\parallel u\parallel_*^p}{p}\in R^+;\Psi(u)\geq\rho\} > A_1(\rho,w),$$

namely

$$\inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u) > \rho\frac{\Phi(w)}{\Psi(w)},$$

and, taking in to account that (i) holds, one has

$$\frac{\inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u)}{\rho} > \frac{\Phi(w) - \inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u)}{\Psi(w) - \rho}.$$

Now, there exists $\lambda \in R$ such that

$$\lambda > \frac{\Phi(w) - \inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u)}{\Psi(w) - \rho}$$

and

$$\lambda < \frac{\inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u)}{\rho},$$

or equivalently

$$\inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u) > \Phi(w) + \lambda(\rho - \Psi(w))$$

and

$$\lambda\rho < \inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u).$$

Therefore, thanks to the $0 < \rho < \Psi(w)$, we obtain

$$\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf\{\Phi(u);\ u \in \Psi^{-1}([\rho,+\infty[)\}, \tag{8}$$

and in other hand,

$$\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) \leq (\Phi(0) + \lambda(\rho - \Psi(0))) = \lambda\rho. \tag{9}$$

So , with (8) and (9), one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf\{\Phi(u);\ u \in \Psi^{-1}([\rho,+\infty[)\}.$$

Therefore, thanks to the

$$\inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))) = \inf_{u\in\Psi^{-1}([\rho,+\infty[)}\Phi(u),$$

we have the

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))). \ \Box$$

Remark 2.2. If in Lemma 2.1, $A_3(\rho, w) - \delta_\rho \leq 0$; the lemma holds again. Because, $A_3(\rho, w) \leq \delta_1 - \delta_2 \leq \delta_1$, and by arguing as in the proof of Lemma 2.1, the results holds.

Here we give an immediate consequence of Lemma 2.1.

Lemma 2.3. Assume that there exist $\rho \in R,\ w \in X$ such that

(i) $0 < \rho < \Psi(w)$,
(ii) $m(\Omega) \sup_{(x,t)\in\Omega\times[-A_3(\rho,w)\ ,\ A_3(\rho,w)]} g(x,t) < \rho$.

where $A_3(\rho, w)$ is given by (6).
Then, there exists $\rho \in R$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))).$$

Proof: Since

$$m(\Omega) \sup_{(x,t)\in\Omega\times[-A_3(\rho,w)+\delta_\rho\ ,\ A_3(\rho,w)-\delta_\rho]} g(x,t) \leq m(\Omega) \sup_{(x,t)\in\Omega\times[-A_3(\rho,w)\ ,\ A_3(\rho,w)]} g(x,t) < \rho,$$

the result holds.
Now, we point out the following result:

Proposition 2.4. The following assertions are equivalent:

(a) there are $\rho \in R,\ w \in X$ such that

(i) $0 < \rho < \Psi(w)$,
(ii) $m(\Omega) \sup_{(x,t)\in\Omega\times[-A_3(\rho,w)\ ,\ A_3(\rho,w)]} g(x,t) < \rho$;

where $A_3(\rho, w)$ is given by (6).
(b) there are $r \in R,\ w \in X$ such that

(j) $0 < r < \Phi(w)$,
(jj) $m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}\sqrt[p]{pr}\ ,\ \frac{k}{c_1}\sqrt[p]{pr}]} g(x,t) < A_2(r, w)$;

where $A_2(r, w)$ is given by (5).

Proof:
$(a) \Rightarrow (b)$. First we note that $0 < \Phi(w)$, because if $\Phi(w) = 0$, one has $\frac{k}{c_1}||w||_* = 0$. Hence, taking into account (ii), one has

$$\Psi(w) \leq m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}||w||_*\ ,\ \frac{k}{c_1}||w||_*\ ]} g(x,t) = 0$$

and that is in contradiction to (i). We now put $A_1(\rho, w) = r$. We obtain $\rho = A_2(r, w)$ and $A_3(\rho, w) = \frac{k}{c_1}\sqrt[p]{pr}$. Therefore, from (i) and (ii), one has

$$0 < r < \Phi(w)$$

and

$$m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}\sqrt[p]{pr}\ ,\ \frac{k}{c_1}\sqrt[p]{pr}]} g(x,t) < A_2(r,w).$$

$(b) \Rightarrow (a)$. First we note that $0 < \Psi(w)$, because if $0 \geq \Psi(w)$, from (j) one has $r\frac{\Psi(w)}{\Phi(w)} \leq 0$; namely, $A_2(r,w) \leq 0$. Hence, from (jj) one has

$$0 = \Psi(0) \leq m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}\sqrt[p]{pr}\ ,\ \frac{k}{c_1}\sqrt[p]{pr}]} g(x,t) < 0,$$

and this is a contradiction. We now put $A_2(r,w) = \rho$. We obtain $r = A_1(\rho,w)$ and $\frac{k}{c_1}\sqrt[p]{pr} = A_3(\rho,w)$. Therefore, from (j) and (jj), we have the conclusion. □

The following lemma is another consequence of Lemma 2.1.

Lemma 2.5. Assume that there exist $r \in R,\ w \in X$ such that

(j) $0 < r < \Phi(w)$,
(jj) $m(\Omega)\sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}\sqrt[p]{pr}\ ,\ \frac{k}{c_1}\sqrt[p]{pr}]} g(x,t) < A_2(r,w)$

where $A_2(r,w)$ is given by (5).
Then, there exists $\rho \in R$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))).$$

Proof: It follows from Lemma 2.3 and Proposition 2.4. □

Finally, we interested in ensuring the existence of at least three weak solutions for the Dirichlet problem (2). Now, we have the following result:

Theorem 2.6. Assume that there exist $\rho \in R$, $b_1 \in L^1(\Omega)$, $w \in X$ and a positive constant γ with $\gamma < p$ such that

(i) $0 < \rho < \int_\Omega g(x, w(x))dx$,
(ii) $m(\Omega)\sup_{(x,t)\in\Omega\times[-A_3(\rho,w)\ ,\ A_3(\rho,w)]} g(x,t) < \rho$
where $A_3(\rho,w)$ is given by (6),
(iii) $g(x,t) \leq b_1(x)(1 + |t|^\gamma)$ almost everywhere in Ω and for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem (2) admits at least three solutions in X whose norms are less than q.

Proof: Our aim is to apply Theorem 1.1. For this end, fix $\lambda \in [0,+\infty[$, and set For each $u \in$X, we put

$$\Phi(u) = \frac{||u||_*^p}{p}$$

and

$$T(u) = -\int_\Omega g(x, u(x))dx.$$

It is well known that T is a continuously Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $T'(u) \in X^*$, given by

$$T'(u)(v) = -\int_\Omega f(x, u(x))v(x)dx$$

for every $v \in X$. Moreover, $T' : X \to X^*$ is a compact operator. Moreover, the functional Φ is continuously Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + a(x)|u(x)|^{p-2}u(x)v(x))dx.$$

Clearly, Φ is a sequentially weakly lower semicontinuous functional and its Gâteaux derivative admits a continuous inverse on X^*.
Thanks to (iii), for each $\lambda > 0$ one has that

$$\lim_{||u||\to+\infty} (\Phi(u) + \lambda T(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$.
Furthermore, thanks to Lemma 2.4, from (i) and (ii) we have

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda T(u) + \lambda\rho) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda T(u) + \lambda\rho).$$

Therefore, we can apply Theorem 1.1. Hence, since the weak solutions of the problem (2) are exactly the solutions of the equation $\Phi'(u) + \lambda T'(u) = 0$, our conclusion follows from Theorem 1.1. $\square$

We also have the following existence result:

Theorem 2.7. Assume that there exist $r \in R$, $b_2 \in L^1(\Omega)$, $w \in X$ and a positive constant γ with $\gamma < p$ such that

(j) $0 < r < \frac{||w||_*^p}{p}$,
(jj) $m(\Omega) \sup_{(x,t)\in\Omega\times[-\frac{k}{c_1}\sqrt[p]{pr}\ ,\ \frac{k}{c_1}\sqrt[p]{pr}]} g(x, t) < A_2(r, w)$
where $A_2(r, w)$ is given by (5),
(jjj) $g(x, t) \leq b_2(x)(1 + |t|^\gamma)$ almost everywhere in Ω and for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem (2) admits at least three solutions in X whose norms are less than q.

Proof: It follows from Lemma 2.5 and Theorem 2.6. □

Remark 2.8. In applying above theorems, it is enough to know an explicit upper bound for the constant k. In particular, if Ω is convex, then the following estimate holds

$$k \leq \frac{N^{-1/p}}{\sqrt{\pi}}[\Gamma(1+\frac{N}{2})]^{1/N}(\frac{p-1}{p-N})^{1-1/p}[m(\Omega)]^{1/N-1/p},$$

and equality occurs when Ω is a ball(see [8], for more details).

We illustrate the results by giving the following example:

Example 2.9. Let $\Omega = \{(x,y) \in R^2; x \geq 0, x^2+y^2 \leq 9\}$ and consider the problem

$$\begin{cases} div(|\nabla u|\nabla u) + \lambda(2u) = x|u|u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem (10) admits at least three solutions in $W_0^{1,3}(\Omega)$ whose norms are less than q. In fact, taking into account $k = \sqrt[6]{\frac{36}{\pi^2}}$ by choosing $\rho = 9\pi$ and $w(x) = \begin{cases} x, & x \in \Omega \\ 0, & e.w \end{cases}$ so that $A_3(\rho, w) = \frac{1}{c_1}\sqrt[6]{\frac{641601}{36}}$, all assumptions of Theorem 2.8, are satisfied with $\gamma = 2$, c_1 is positive constant such that the inequality (3) holds for $m(x) = x$ and η sufficiently large, also with choose $r = \pi$ so that $A_2(r, w) = \frac{324}{801}\pi$, all assumptions of Theorem 2.9, are satisfied with μ sufficiently large.

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