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New type of multivalued F-contraction involving fixed points on closed ball

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Abstract

This paper is a continuation of the investigations of F-contraction. The aim of this article is to extend the concept of F-contraction on closed ball. We introduce the notion of Ćirić type multivalued F-contraction on closed ball and establish new fixed point theorems for Ćirić type multivalued F-contraction on closed ball in a complete metric space. Our results are very useful for the contraction of the mapping only on closed ball instead on the whole space. Some comparative examples are constructed whose illustrate the superiority of our results. Our results provide extension as well as substantial generalizations and improvements of several well-known results in the existing comparable literature.

Keywords: Metric space, fixed point, F-contraction, closed ball.
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1. Introduction

We recollect some essential notations, required definitions, and primary results coherent with the literature. For a nonempty set X, we denote by $N(X)$ the class of all nonempty subsets of X. Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $D(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $K(X)$ the class of all compact subsets of X. Let H be the Hausdorff metric induced by the metric d on X, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B),\ \sup_{y\in B} D(y, A)\right\},$$

for every $A, B \in CB(X)$. Let $T : X \longrightarrow CB(X)$ be a multi-valued mapping. A point $q \in X$ is said to be a fixed point of T if $q \in Tq$.

The fixed point theory of multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [35], who extended the Banach contraction principle to multivalued mappings.

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Theorem 1.1 ([35]). *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow CB(X)$ *be a multi-valued mapping such that for all* $x, y \in X$

$$H(Tx, Ty) \leqslant kd(x, y),$$

where $0 < k < 1$. *Then* T *has a fixed point.*

From the application point of view the situation is not yet completely satisfactory because there are many situations in which the mappings are not contractive on the whole space but instead they are contractive on its subsets. However, by imposing a subtle restriction, one can establish the existence of a fixed point of such mappings. Shoaib et al. [41] proved significant results concerning the existence of fixed points of the dominated self-mappings satisfying some contractive conditions on closed ball in a 0-complete quasi-partial metric space. Other results on closed ball can be seen in [9–12]. Over the years, fixed point theory has been generalized in different ways by several mathematicians (see [2, 3, 8, 13–19, 23–25, 29, 31, 33, 40].

For $x \in X$ and $\varepsilon > 0$, $\overline{B(x, \varepsilon)} = \{y \in X : d(x, y) \leqslant \varepsilon\}$ is a closed ball in (X, d).

Definition 1.2 ([38]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geqslant 1$ implies that $\alpha(Tx, Ty) \geqslant 1$.

Definition 1.3 ([37]). Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geqslant \eta(x, y)$ implies that $\alpha(Tx, Ty) \geqslant \eta(Tx, Ty)$.

If $\eta(x, y) = 1$, then above definition reduces to Definition 1.2. If $\alpha(x, y) = 1$, then T is called an η-subadmissible mapping.

Definition 1.4 ([26]). Let (X, d) be a metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-η-continuous mapping on (X, d) if for given $x \in X$, and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty,\ \alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

Hussain et al. [27] modified the notions of α_*-admissible and α_*-ψ-contractive mappings as follows:

Definition 1.5 ([27]). Let $T : X \to 2^X$ be a multifunction, $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions where η is bounded. We say that T is α_*-admissible mapping with respect to η if $\alpha(x, y) \geqslant \eta(x, y)$ implies $\alpha_*(Tx, Ty) \geqslant \eta_*(Tx, Ty)$, $x, y \in X$, where $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A,\ y \in B\}$ and $\eta_*(A, B) = \sup\{\eta(x, y) : x \in A,\ y \in B\}$.

If $\eta(x, y) = 1$ for all $x, y \in X$, then this definition reduces to [27, Definition 4.1]. In Definition 1.5, if $\alpha(x, y) = 1$ for all $x, y \in X$, then T is called η_*-subadmissible mapping.

Definition 1.6 ([33]). Let (X, d) be a metric space. Let $T : X \to CL(X)$ and $\alpha : X \times X \to [0, +\infty)$ be two functions. We say that T is α-continuous multivalued mapping on $(CL(X), H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $\lim\limits_{n \longrightarrow \infty} d(x_n, x) = 0$, $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ we have $\lim\limits_{n \longrightarrow \infty} H(Tx_n, Tx) = 0$.

In 2012, Wardowski [45] introduced a new type of contractions called F-contraction and proved new fixed point theorems concerning F-contraction. He generalized the Banach contraction principle in a different way than as it was done by different investigators; see [1, 4, 6, 7, 20–22, 28, 32, 34, 39, 40, 42–44]. Piri et al. [36] defined the F-contraction as follows:

Definition 1.7 ([36]). Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leqslant F\left(d(x, y)\right), \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $\kappa \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by Δ_F, the set of all functions satisfying the conditions (F1)-(F3).

Example 1.8 ([45]). Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies (F1)-(F3) for any $\kappa \in (0,1)$. Each mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(Tx, Ty) \leqslant e^{-\tau} d(x, y) \quad \text{for all } x, y \in X,\ Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leqslant e^{-\tau} d(x, y)$, also holds, i.e., T is a Banach contraction.

Example 1.9 ([45]). If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$, then F satisfies (F1)-(F3) and the condition (1.1) is of the form

$$\frac{d(Tx, Ty)}{d(x, y)} \leqslant e^{d(Tx,Ty)-d(x,y)} \leqslant e^{-\tau} \text{ for all } x, y \in X,\ Tx \neq Ty.$$

Remark 1.10. From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [45] stated a modified version of the Banach contraction principle as follows.

Theorem 1.11 ([45]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Hussain et al. [26] introduced the following family of new functions.

Let Δ_G denote the set of all functions $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ satisfying: (G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1t_2t_3t_4 = 0$, there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Definition 1.12 ([26]). Let (X, d) be a metric space and T be a self-mapping on X. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-η-GF-contraction if for $x, y \in X$, with $\eta(x, Tx) \leqslant \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leqslant F(d(x, y)),$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

Lately, Acar et al. [5] introduced the concept of generalized multivalued F-contraction mappings and established a fixed point result, which was a proper generalization of some multivalued fixed point theorems including Nadler's.

Definition 1.13 ([5]). Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that

$$x, y \in X,\ H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leqslant F(M(x, y)),$$

where

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\}.$$

Theorem 1.14 ([5]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a generalized multivalued F-contraction. If T or F is continuous, then T has a fixed point in X.*

We now introduce the concept of α-η-continuous for multivalued mappings in metric spaces.

Definition 1.15. Let (X, d) be a metric space. Let $T : X \to CB(X)$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-η-continuous multivalued mapping on $(CB(X), H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $x_n \overset{d}{\to} x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ we have $Tx_n \overset{H}{\to} Tx$, that is $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ we have $\lim_{n\to\infty} H(Tx_n, Tx) = 0$.

The following result regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball is given in [30, Theorem 5.1.4]. The result is very useful in the sense that it requires the contraction of the mapping only on the closed ball instead on the whole space.

Theorem 1.16 ([30]). *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$, and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(Tx, Ty) \leqslant kd(x, y) \textit{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, Tx_0) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$.

2. Fixed point theorem for Ćirić type GF-contraction on closed ball

In this section, we introduce multivalued fixed point theorem for modified F-contraction on closed ball in complete metric spaces. We define multivalued α-η-GF-contraction on closed ball as follows:

Definition 2.1. Let (X, d) be a metric space and $T : X \to CB(X)$. Also suppose that $\alpha, \eta : X \times X \to \{-\infty\} \cup (0, +\infty)$ are two functions. We say that T is Ćirić type multivalued α-η-GF-contraction on closed ball if for $x, y \in \overline{B(x_0, r)} \subseteq X$ with $\eta_*(x, Tx) \leqslant \alpha_*(x, y)$ and $Tx \neq Ty$ we have

$$2G\left(D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\right) + F\left(H(Tx, Ty)\right) \leqslant F\left(M\left(x, y\right)\right),$$

where

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\},$$

and

$$D(x_0, Tx_0) \leqslant (1 - k)\, r,$$

where $k \in [0, 1)$, $G \in \Delta_G$, and $F \in \Delta_F$.

Now we state our main result.

Theorem 2.2. *Let (X, d) be a complete metric space. Suppose $T : X \to CB(X)$ is Ćirić type multivalued α-η-GF-contraction on closed ball $\overline{B(x_0, r)}$ satisfying the following assertions:*

(i) T is an α_*-admissible mapping with respect to η;
(ii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geqslant \eta_*(x_0, Tx_0)$;
(iii) $\sum_{j=0}^{\mathbb{N}} D(x_0, Tx_0) \leqslant r$ for all $j \in \mathbb{N}$;
(iv) T is α-η-continuous multivalued mapping.

Then there exists a fixed point x^* in $\overline{B(x_0, r)}$ such that $x^* \in Tx^*$.

Proof. Let $x_0 \in X$, such that $\alpha_*(x_0, Tx_0) \geqslant \eta_*(x_0, Tx_0)$. Since T is an α_*-admissible mapping with respect to η then there exists $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha_*(x_0, Tx_0) \geqslant \eta_*(x_0, Tx_0) = \eta(x_0, x_1).$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So, we assume that $x_0 \neq x_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\left(hH\left(Tx_0, Tx_1\right)\right) < F\left(H\left(Tx_0, Tx_1\right)\right) + G\left(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)\right).$$

Now from $D(x_1, Tx_1) < hH(Tx_0, Tx_1)$, we deduce that there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leqslant hH(Tx_0, Tx_1)$. Consequently, we obtain

$$F(D(x_1, Tx_1)) \leqslant F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)),$$

which implies

$$\begin{aligned}
&2G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)) + F(D(x_1, x_2)) \\
&\quad \leqslant 2G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)) + F(H(Tx_0, Tx_1)) \\
&\qquad + G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)) \\
&\quad \leqslant F(M(x_0, x_1)) + G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)),
\end{aligned}$$

and hence

$$\begin{aligned}
&G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), 0) + F(M(Tx_0, Tx_1)) \\
&\quad \leqslant F\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2}\right\}\right).
\end{aligned} \tag{2.1}$$

Now, since $d(x_0, x_1).d(x_1, x_2).d(x_0, x_2).0 = 0$, so from (G) there exists $\tau > 0$ such that

$$G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), 0) = \tau.$$

Therefore from (2.1) we deduce that

$$\begin{aligned}
\tau + F(d(x_1, Tx_1)) &\leqslant F\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2}\right\}\right) \\
&= F\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2}\right\}\right) - \tau \\
&= F\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_0, x_2) + D(x_1, x_1)}{2}\right\}\right) - \tau \\
&\leqslant F(\max\{D(x_0, Tx_0), D(x_1, Tx_1)\}) - \tau.
\end{aligned} \tag{2.2}$$

If $\max\{D(x_0, Tx_0), D(x_1, Tx_1)\} = D(x_1, Tx_1)$, then (2.2) becomes

$$F(D(x_1, Tx_1)) \leqslant F(D(x_1, Tx_1)) - \tau,$$

which does not hold true. Thus $\max\{D(x_0, Tx_0), D(x_1, Tx_1)\} = D(x_0, Tx_0)$. Consequently,

$$F(D(x_1, Tx_1)) \leqslant F(D(x_0, Tx_0)) - \tau.$$

From (iii), we deduce that

$$d(x_0, x_1) = D(x_0, Tx_0) \leqslant r,$$

thus, $x_1 \in \overline{B(x_0, r)}$. Suppose $x_2 .... x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. As F is strictly increasing and repeating these steps for $x_2 .... x_j$, we obtain

$$D(x_j, Tx_j) \leqslant D(x_{j-1}, Tx_{j-1}). \tag{2.3}$$

Now, using triangular inequality and (2.3), we get

$$d(x_0, x_{j+1}) \leqslant d(x_0, x_1) + ... + d(x_j, x_{j+1}) \leqslant \sum_{j=0}^{\mathbb{N}} D(x_0, Tx_0) \leqslant r.$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. By continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$,

$$\eta(x_{n-1}, x_n) = \eta_*(x_{n-1}, Tx_{n-1}) \leqslant \alpha_*(x_{n-1}, Tx_{n-1}) = \alpha(x_{n-1}, x_n),$$

and

$$\begin{aligned}
&\tau + F(d(x_n, x_{n+1}))\\
&\quad \leqslant F\left(\max\left\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2}\right\}\right)\\
&\quad = F\left(\max\left\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n)}{2}\right\}\right) - \tau\\
&\quad \leqslant F(\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\}) - \tau.
\end{aligned}$$

If $\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} = D(x_n, Tx_n)$, then

$$F(D(x_n, Tx_n)) \leqslant F(D(x_n, Tx_n)) - \tau.$$

Thus, $\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} = D(x_{n-1}, Tx_{n-1})$, we obtain

$$F(d(x_n, x_{n+1})) \leqslant F(D(x_{n-1}, Tx_{n-1})) - \tau, \tag{2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (2.4), we have

$$F(d(x_n, x_{n+1})) \leqslant F(D(x_{n-1}, Tx_{n-1})) - \tau \leqslant F(D(x_{n-2}, Tx_{n-2})) - 2\tau \leqslant \cdots \leqslant F(D(x_0, Tx_0)) - n\tau, \tag{2.5}$$

for all $n \in \mathbb{N}$. Since $F \in \Delta_F$, so by taking limit as $n \longrightarrow \infty$ in (2.5), we deduce

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty \iff \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Now from (F3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) = 0. \tag{2.7}$$

By (2.4), we have

$$\begin{aligned}
&d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) - d(x_n, x_{n+1})^k F(d(x_0, x_1))\\
&\quad \leqslant d(x_n, x_{n+1})^k [F(d(x_0, x_1) - n\tau)] - d(x_n, x_{n+1})^k F(d(x_0, x_1))\\
&\quad = -n\tau [d(x_n, x_{n+1})]^k \leqslant 0.
\end{aligned} \tag{2.8}$$

Letting $n \longrightarrow \infty$ in (2.8) and applying (2.6) and (2.7), we have,

$$\lim_{n\longrightarrow\infty} n [d(x_n, x_{n+1})]^k = 0, \tag{2.9}$$

we observe that from (2.9), then there exists $n_1 \in \mathbb{N}$, such that $n(d(x_n, x_{n+1}))^k \leqslant 1$ for all $n \geqslant n_1$, we get

$$d(x_n, x_{n+1}) \leqslant \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geqslant n_1. \tag{2.10}$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. Then, by the triangle inequality and from (2.10) we have

$$\begin{aligned}
d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m)\\
&= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. This implies that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x^* \in X$ such that $\lim_{n\longrightarrow\infty} d(x_n, x^*) = 0$. By (2.3) and α-η-continuity of the multi-valued mapping T, we get

$$\lim_{n\to\infty} H(Tx_n, Tx^*) = 0.$$

Now we obtain

$$D(x^*, Tx^*) = \lim_{n\to\infty} D(x_{n+1}, Tx^*) \leqslant \lim_{n\to\infty} H(Tx_n, Tx^*) = 0.$$

Therefore, $x^* \in Tx^*$ and hence T has a fixed point. □

Example 2.3. Let $X = \mathbb{R}^+$. Define $T : X \to CB(X)$, $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$, $\eta : X \times X \to \mathbb{R}^+$, $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$, and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$Tx = \begin{cases} \left[0, \frac{x}{3}\right], & \text{if } x \in [0,1], \\ 2x, & \text{if } x \in (1, \infty), \end{cases} \qquad \alpha(x,y) = \begin{cases} e^{x+y}, & \text{if } x \in [0,1], \\ \frac{1}{3}, & \text{otherwise}, \end{cases}$$

$$\eta(x,y) = \frac{1}{2} \text{ for all } x, y \in X, \qquad G(t_1, t_2, t_3, t_4) = \tau > 0 \text{ and } F(t) = \ln t \text{ with } t > 0.$$

And $x_0 = \frac{1}{2}, r = 1, \overline{B(x_0, r)} = [0,1]$, then

$$d(\frac{1}{2}, T\frac{1}{2}) = \left|\frac{1}{2} - \frac{1}{6}\right| = \frac{1}{3} < r.$$

If $x, y \in \overline{B(x_0, r)}$, then $\alpha(x,y) = e^{x+y} \geqslant \frac{1}{2} = \eta(x,y)$. On the other hand, $Tx \in [0,1]$ for all $x \in [0,1]$. Then $\alpha(Tx, Ty) \geqslant \eta(x, Tx)$ with $H(Tx, Ty) = \left|\frac{x}{3} - \frac{y}{3}\right| > 0$ and clearly $\alpha(0, T0) \geqslant \eta(0, T0)$. Hence we have

$$H(Tx, Ty) = \left|\frac{x}{3} - \frac{y}{3}\right| < |x - y| \leqslant M(x,y).$$

Consequently,

$$\tau + F\left(H(Tx, Ty)\right) = \tau + \ln H(Tx, Ty) \leqslant \ln M(x,y) = F\left(M(x,y)\right).$$

If $x \notin \overline{B(x_0, r)}$ or $y \notin \overline{B(x_0, r)}$, then $\alpha(x,y) = \frac{1}{3} \ngeq \frac{1}{2} = \eta(x,y)$, either

$$\begin{aligned} 2|x-y| &> |x-y|, \\ |2x - 2y| &> |x-y|, \\ |Tx - Ty| &> |x-y|, \\ \tau + F\left(d(Tx, Ty)\right) &\geqslant F\left(d(x,y)\right). \end{aligned}$$

Then the contractive condition does not hold on X.

3. Conclusion

In this connection, the main aim of our paper is to present fixed point theorem for modified F-contraction on closed ball for multivalued mapping and different from F-contractions given in [26, 36, 45]. Existence of fixed point results of such type of F-contraction on closed ball in complete metric space are established. The study of results is very useful in the sense that it requires the F-contraction mapping only on the closed ball instead on the whole space. The new concepts lead to further investigations and applications. It will be also interesting to apply these concepts in a different metric spaces.

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