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Numerical Solution of 12th Order Boundary Value Problems by Using Homotopy Perturbation Method

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ABSTRACT

In this paper, a homotopy-perturbation method (HPM) [1-6, 26-28] is used to solve both linear and nonlinear Nth boundary value problems with two point boundary conditions for ninth-order, tenth-order and twelfth-order. By applying (HPM) on three examples the numerical results are compared with the exact solution, to show effectiveness and accuracy of the method.

Keywords: HPM – linear and non-linear problems – boundary value problem – Approximate solution

1. Introduction

A new perturbation method called homotopy perturbation method (HPM) was proposed by He in 1997 and systematical description in 2000 which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [1-2]. This new method was further developed and improved by He and applied to nonlinear oscillators with discontinuities [3], nonlinear wave equations [4], asymptotology [5], boundary value problem [6], limit cycle and bifurcation of nonlinear problems [7] and many other subjects. Thus He's method is a universal one which can solve various kinds of nonlinear equations. For example, it was applied to the quadratic Ricatti differential equation by Abbasbandy [8]; to the axisymmetric flow over a stretching sheet by Ariel et al. [9]; to the nonlinear systems of reaction-diffusion equations by Ganji and Sadighi [10]; to the Helmholtz equation and fifth-order KdV equation by Rafei and Ganji [11]; for the thin film flow of a fourth grade fluid down a vertical cylinder by Siddiqui et al. [12]; to the nonlinear Voltra-Fredholm integral equations by Ghasemi et al. [13].

In recent years, the application of the homotopy perturbation method (HPM) [5, 7, 30] in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problem under study into a simple problem which is easy to solve. Most perturbation methods assume a small parameter exists, but most nonlinear problem have no small at all. Many new methods, such as the variational method [36-37], variational iterations method [14, 17] and (DTM) [38-39].

The homotopy perturbation method proposed by He [2, 26] is constantly being developed and applied to solve various nonlinear problems [3, 4, 7, 11, 27-32, 34]. Unlinke analytical perturbation methods, the homotopy-perturbation method does not depend on a small parameter which is difficult to find. We focus on dealing with the N^{th} boundary value problems by means of the homotopy perturbation method. Three numerical examples will be presented to verify the homotopy perturbation method.

Recently various powerful mathematical methods such as variational iteration method [14-23], exp-function method [24-25], f-expansion method [24], this paper, applies the homotopy perturbation method (HPM) [1-5].

2. Homotopy perturbation method

We consider the general N^{th} order boundary value problems of the type:

$$y^{(2n)}(x) = f(x, y, y'(x), \ldots, y^{(2n-1)}(x)), \quad 0 < x < 1 \tag{1}$$

with the boundary conditions

$$y^{(2k)}(0) = \alpha_{2k}, \quad k = 0,1,2,3\cdots,(n-1) \tag{2}$$

$$y^{(2k)}(1) = \beta_{2k}, \quad k = 0,1,2,3\cdots,(n-1) \tag{3}$$

where $f(x, y, y'(x), \ldots, y^{(2n-1)}(x))$ and $y(x)$ are assumed real and as many as times differentiable as required for $x \in [0,1]$, α_{2k} and β_{2k}, $k = 0,1,2,\cdots,(n-1)$ are real finite constants Djidjedi, Twizell and Boutayeb [40], more over the constants α_{2k}, $k = 0,1,2,\cdots,(n-1)$ describe the even order derivatives at the boundary $x = 0$.

Using the transformation

$$y = y_1,\ \frac{dy}{dx} = y_2,\ \frac{d^2y}{dx^2} = y_3, \cdots,\ \frac{d^{2n-1}y}{dx^{2n-1}} = y_{2n} \tag{4}$$

We can rewrite the N^{th}-order boundary value problems (1), (2) and (3) as the system of ordinary differential equations:

$$\begin{aligned}
&\frac{dy}{dx} = y_2,\\
&\frac{d^2y}{dx^2} = y_3,\\
&\vdots\\
&\frac{d^{2n-1}y}{dx^{2n-1}} = y_{2n},\\
&\frac{dy}{dx} = f(x, y, y_1(x), y_2(x), \cdots, y_{2n-1}(x)).
\end{aligned} \tag{5}$$

with the boundary conditions

$$y_1(a) = \alpha_0, y_2(a) = \alpha_1, \cdots, y_{2n}(a) = \alpha_{2n-1} \tag{6}$$

or

$$y_1(b) = \beta_0, y_2(b) = \beta_1, \cdots, y_{2n}(b) = \beta_{2n-1} \tag{7}$$

which can be written as a system of integral equations:

$$\begin{aligned}
&y_1 = \alpha_0 + \int_0^x y_2(t)\,dt,\\
&y_2 = \alpha_1 + \int_0^x y_3(t)\,dt,\\
&y_3 = \alpha_2 + \int_0^x y_4(t)\,dt,\\
&\vdots\\
&y_{2n} = \alpha_{2n-1} + \int_0^x f(t, y_1(t), y_2(t), \cdots, y_{2n-1}(t))\,dt.
\end{aligned} \tag{8}$$

To explain (HPM), we consider (8) as [2, 27]:

$$L(y_1,y_2,\cdots,y_{2n})=\begin{bmatrix} L_1(y_1,y_2,\cdots,y_{2n} \\ \vdots \\ L_n(y_1,y_2,\cdots,y_{2n}\end{bmatrix}=0, \tag{9}$$

with solution $(f_1,f_2,\cdots,f_{2n})$ where

$$L(y_1,y_2,\cdots,y_{2n})=y_1-\alpha_0-\int_0^x y_2(t)dt, \qquad L_{2n}(y_1,y_2,\cdots,y_{2n})=y_{2n}-\alpha_{2n-1}-\int_0^x f(t,y_1(t),y_2(t),\cdots,y_{2n-1}(t))dt.$$
$$\vdots \tag{10}$$

We can define homotopy $H(y_1,y_2,\cdots,y_{2n},p)$ by

$$H(y_1,y_2,\cdots,y_{2n},0)=F(y_1,y_2,\cdots,y_{2n},H(y_1,y_2,\cdots,y_{2n},1)$$
$$=L(y_1,y_2,\cdots,y_{2n}),$$

where

$$F(y_1,y_2,\cdots,y_{2n})=[F(y_1,y_2,\cdots,y_{2n}),\cdots,F_{2n}(y_1,y_2,\cdots,y_{2n})]^T$$
$$=[y_1-\alpha_0,\cdots,y_{2n-1}-\alpha_{2n-2},y_{2n}-\alpha_{2n-1}]^T. \tag{11}$$

$$H(y_1,y_2,\cdots,y_{2n},p)=[H_1(y_1,y_2,\cdots,y_{2n},p),\cdots,H_n(y_1,y_2,\cdots,y_{2n},p)]^T. \tag{12}$$

Typically we may choose a convex homotopy by:

$$H(y_1,y_2,\cdots,y_{2n},p)=(1-p)F(y_1,y_2,\cdots,y_{2n})+pL(y_1,y_2,\cdots,y_{2n})=0. \tag{13}$$

The convex homotopy (13) contiuously trace an implicitly defined carve from a starting point $H(y_1-\alpha_0,y_2-\alpha_2\cdots,y_{2n}-\alpha_{2n-1},0)$ to a solution function $H(f_1,f_2,\cdots,f_{2n},1)$. The embedding parameter p monotonically increasing from zero to unite as trivial problem $F(y_1,y_2,\cdots,y_{2n})=0$ is continuously deformed to original problem $L(y_1,y_2,\cdots,y_{2n})=0$.

The (HPM) uses the homotopy parameter p as an expanding parameter:

$$y_1=y_{10}+py_{11}+p^2y_{12}+\cdots,$$
$$\vdots$$
$$y_{2n}=y_{2n0}+py_{2n1}+p^2y_{2n2}+\cdots. \tag{14}$$

The approximate solution of Eq. (1), therefor, can be readily obtained:

$$f_1=\lim_{p\to 1}y_1=y_{10}+y_{11}+y_{12}+\cdots,$$
$$\vdots$$
$$f_{2n}=\lim_{p\to 1}y_{2n}=y_{2n0}+y_{2n1}+y_{2n2}+\cdots. \tag{15}$$

The convergence of the series (14) for the application of (HPM) to (8), we can write Eq. (13) as follows:

$$H_1(y_1,y_2,\cdots,y_{2n},p)=(1-p)F_1(y_1,y_2,\cdots,y_{2n})+pL_1(y_1,y_2,\cdots,y_{2n})=0,$$
$$\vdots$$
$$H_{2n}(y_1,y_2,\cdots,y_{2n},p)=(1-p)F_{2n}(y_1,y_2,\cdots,y_{2n})+pL_{2n}(y_1,y_2,\cdots,y_{2n})=0. \tag{16}$$

substitution of (10), (11) and (13) into (16) yields

$$(1-p)(y_1-\alpha_0)+p(y_1-\alpha_0-\int_0^x y_2(t)dt=0,$$
$$\vdots$$
$$(1-p)(y_{2n}-\alpha_{2n-1})+p(y_{2n}-\alpha_{2n-1}-\int_0^x f(t,y_1(t),y_2(t),\cdots,y_{2n-1}(t))dt=0. \tag{17}$$

By equating the terms with identical powers of p, we have

$$p^0:\begin{cases} y_{10}=\alpha_0, \\ y_{20}=\alpha_1, \\ \vdots \\ y_{2n0}=\alpha_{2n-1}, \end{cases} \tag{18}$$

$$p^1:\begin{cases} y_{11}=\int_0^x y_{20}(t)dt, \\ \vdots \\ y_{2n-1,1}=\int_0^x y_{2n0}(t)dt, \\ y_{2n,1}=\int_0^x \frac{d}{dp}(f(t,y_1(t),\cdots,y_{2n-2}(t),y_{2n-1}(t)))dt, \end{cases} \tag{19}$$

$$p^2:\begin{cases} y_{12}=\int_0^x y_{21}(t)dt, \\ \vdots \\ y_{2n-1,2}=\int_0^x y_{2n1}(t)dt, \\ y_{2n,2}=\int_0^x \frac{d^2}{dp^2}(f(t,y_1(t),\cdots,y_{2n-2}(t),y_{2n-1}(t)))dt, \end{cases} \tag{20}$$

$$\vdots$$

$$p^{2k}:\begin{cases} y_{1,2k}=\int_0^x y_{2,2k-1}(t)dt, \\ \vdots \\ y_{2n-1,2k}=\int_0^x y_{2n,2k-1}(t)dt, \\ y_{2n,2k}=\int_0^x \frac{d^{2k}}{dp^{2k}}(f(t,y_1(t),\cdots,y_{2n-2}(t),y_{2n-1}(t)))dt. \end{cases} \tag{21}$$

Combining all the terms of Eqs. (18)-(21) give the solution of the problem, by using the boundary conditions (6) and (7) we can obtained all parameters.

3. Applications

In this section, in order to verify numerically whether the proposed methodology leads to higher accuracy, we evaluate the numerical solution of the problem. To show the efficiency of the present method for our problem in comparison with the exact solution we report absolute error which is defined by

$$E_{yN}(x)=\mathrm{abs}(y_{Exact}(x)-y^N{}_{Approx}(x)). \tag{22}$$

where

$$y^N{}_{Approx}(x)=\sum_{k=0}^{N} y_{1k} \qquad \text{for} \quad N=0,1,2,\cdots . \tag{23}$$

4. Examples

Example 1. Consider the following linear ninth-order problem

$$y^{(9)}(x)=y(x)-9e^x, \ a<x<b. \tag{24}$$

with the following boundary conditions

$$y^{(k)}(0) = (1-k), \quad k = 0,1,2,3,4. \tag{25}$$

$$y^{(k)}(1) = (-ke), \quad k = 0,1,2,3. \tag{26}$$

The exact solution is

$$y(x) = (1-x)e^{x}. \tag{27}$$

Using the transformation (4) we can rewrite the ninth-order boundary value problem (24) as the system of integral equations:

$$\begin{aligned}
y_1 &= 1 + \int_0^x y_2(t)\,dt, \\
y_2 &= \int_0^x y_3(t)\,dt, \\
y_3 &= -1 + \int_0^x y_4(t)\,dt, \\
y_4 &= -2 + \int_0^x y_5(t)\,dt, \\
y_5 &= -3 + \int_0^x y_6(t)\,dt, \\
y_6 &= a + \int_0^x y_7(t)\,dt, \\
y_7 &= b + \int_0^x y_8(t)\,dt, \\
y_8 &= c + \int_0^x y_9(t)\,dt, \\
y_9 &= d + \int_0^x [-9e^t + y_1(t)]\,dt.
\end{aligned} \tag{28}$$

Using (17) into (28) we have

$$\begin{aligned}
&y_{10} + p y_{11} + p^2 y_{12} + \cdots = 1 + p\int_0^x (y_{20} + p y_{21} + p^2 y_{22} + \cdots)\,dt, \\
&y_{20} + p y_{21} + p^2 y_{22} + \cdots = p\int_0^x (y_{30} + p y_{31} + p^2 y_{32} + \cdots)\,dt, \\
&\vdots \\
&y_{90} + p y_{91} + p^2 y_{92} + \cdots = d + p\int_0^x (-9e^t + y_{10} + p y_{11} + p^2 y_{12} + \cdots)\,dt.
\end{aligned} \tag{29}$$

Comparing the coefficients of like powers of p, we have:

$$p^0:\begin{cases} y_{10}=1,\\ y_{20}=0,\\ y_{30}=-1,\\ y_{40}=-2,\\ y_{50}=-3,\\ y_{60}=a,\\ y_{70}=b,\\ y_{80}=c,\\ y_{90}=d. \end{cases} \tag{30}$$

$$p^1:\begin{cases} y_{11}=0,\\ y_{21}=-x,\\ y_{31}=-2x,\\ y_{41}=-3x,\\ y_{51}=ax,\\ y_{61}=bx,\\ y_{71}=cx,\\ y_{81}=dx,\\ y_{91}=-9e^{x}+x+9. \end{cases} \tag{31}$$

$$p^2:\begin{cases} y_{12}=-\frac{1}{2}x^2,\\ y_{22}=-x^2,\\ y_{32}=-\frac{3}{2}x^2,\\ y_{42}=\frac{1}{2}ax^2,\\ y_{52}=\frac{1}{2}bx^2,\\ y_{62}=\frac{1}{2}cx^2,\\ y_{72}=\frac{1}{2}dx^2,\\ y_{82}=-9e^{x}+\frac{1}{2}x^2+9x+9,\\ y_{92}=0. \end{cases} \tag{32}$$

$$\vdots$$

$$p^8:\begin{cases} y_{18}=\frac{1}{40320}dx^8, \\ y_{28}=-9e^x+\frac{1}{40320}dx^8+\frac{1}{480}x^7+\frac{1}{40}x^6+\frac{3}{20}x^5+\frac{3}{8}x^4 \\ \quad +\frac{3}{2}x^3+\frac{9}{2}x^2+9x+9, \\ y_{38}=0, \\ y_{48}=-\frac{1}{40320}x^8, \\ y_{58}=-\frac{1}{40320}x^8, \\ y_{68}=-\frac{1}{13440}x^8, \\ y_{78}=\frac{1}{40320}ax^8, \\ y_{88}=\frac{1}{40320}bx^8, \\ y_{98}=\frac{1}{40320}cx^8. \end{cases} \tag{33}$$

$$p^9:\begin{cases} y_{19}=-9e^x+\frac{1}{362880}x^9+\frac{1}{3840}x^8+\frac{1}{280}x^7+\frac{1}{40}x^6+\frac{3}{40}x^5 \\ \quad +\frac{3}{8}x^4+\frac{3}{2}x^3+\frac{9}{2}x^2+9x+9, \\ y_{29}=0, \\ y_{39}=-\frac{1}{362880}x^9, \\ y_{49}=-\frac{1}{362880}x^9, \\ y_{59}=-\frac{1}{120960}x^9, \\ y_{69}=\frac{1}{362880}ax^9, \\ y_{79}=\frac{1}{362880}bx^9, \\ y_{89}=\frac{1}{362880}cx^9, \\ y_{99}=\frac{1}{362880}dx^9. \end{cases} \tag{34}$$

Combining all the terms of Eqs. (30)-(34) gives

$$y^{(9)}(x)=1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4+\frac{1}{80}x^6+\frac{1}{26880}x^8-\frac{1}{45360}x^9+\frac{1}{120}ax^5 \\ +\frac{1}{720}bx^6+\frac{1}{5040}cx^7+\frac{1}{40320}dx^8. \tag{35}$$

Using the boundary conditions (26) then we have:

$$a=-3.998498472,\quad b=-14.03378034,$$
$$c=-14.68710863,\quad d=-9.727056391.$$

$$y^{(9)}(x)=1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4-0.03332082060x^5-0.00699136158x^6 \\ -0.001128394569x^7-0.0002040440574x^8-\frac{1}{45360}x^9. \tag{36}$$

The numerical results obtained in Table 1.1. In Table 1.1, we list the results obtained by homotopy perturbation method (HPM) and compared with the exact solution. As we see from this Table, it is clear that the results obtained by the present method are very superior to that obtained by the exact solution highly accurate.

Table 1.1:

Linear ninth-order BVP

X	HPM(N=9)	$y_{exact} = (1-x)e^{x}$
0.0	0.100000000E+01	0.100000000E+01
0.2	0.9771222083E+00	0.9771222064E+00
0.4	0.8950948366E+00	0.8950948188E+00
0.6	0.7288475468E+00	0.7288475200E+00
0.8	0.4451081901E+00	0.4451081856E+00
1.0	0.0000000000E+00	0.0000000000E+00

Example 2. Consider the following linear tenth-order problem

$$y^{(10)}(x) = e^{-x}y^{2}(x), \quad a < x < b. \tag{37}$$

with the following boundary conditions

$$y^{(2k)}(0) = (1), \quad k = 0,1,2,3,4. \tag{38}$$

$$y^{(2k)}(1) = (e), \quad k = 0,1,2,3,4. \tag{39}$$

The exact solution is

$$y(x) = e^{x}. \tag{40}$$

Using the transformation (4) we can rewrite the tenth-order boundary value problem (37) as the system of integral equations:

$$\begin{aligned}
y_1 &= 1 + \int_0^x y_2(t)dt,\\
y_2 &= a + \int_0^x y_3(t)dt,\\
y_3 &= 1 + \int_0^x y_4(t)dt,\\
y_4 &= b + \int_0^x y_5(t)dt,\\
y_5 &= 1 + \int_0^x y_6(t)dt,\\
y_6 &= c + \int_0^x y_7(t)dt,\\
y_7 &= 1 + \int_0^x y_8(t)dt,\\
y_8 &= d + \int_0^x y_9(t)dt,\\
y_9 &= 1 + \int_0^x y_{10}(t)dt,\\
y_{10} &= f + \int_0^x [e^{-t}y_1^{(2)}(t)]dt.
\end{aligned} \tag{41}$$

Using (17) into (41) we have

$$y_{10}+py_{11}+p^2y_{12}+\cdots=1+p\int_0^x(y_{20}+py_{21}+p^2y_{22}+\cdots)dt,$$

$$y_{20}+py_{21}+p^2y_{22}+\cdots=a+p\int_0^x(y_{30}+py_{31}+p^2y_{32}+\cdots)dt,$$

$$\vdots$$

$$y_{100}+py_{101}+p^2y_{102}+\cdots=f+p\int_0^x e^{-t}(y_{10}+py_{11}+p^2y_{12}+\cdots)^2dt. \quad (42)$$

Comparing the coefficients of like powers of p, we have:

$$p^0:\begin{cases} y_{10}=1,\\ y_9=a,\\ y_{30}=1,\\ y_{40}=b,\\ y_{50}=1,\\ y_{60}=c,\\ y_{70}=1,\\ y_{80}=d,\\ y_{90}=1,\\ y_{100}=f. \end{cases} \quad (43)$$

$$p^1:\begin{cases} y_{11}=ax,\\ y_{21}=x,\\ y_{31}=bx,\\ y_{41}=x,\\ y_{51}=cx,\\ y_{61}=x,\\ y_{71}=dx,\\ y_{81}=x,\\ y_{91}=fx,\\ y_{101}=-e^{-x}+1. \end{cases} \quad (44)$$

$$p^2:\begin{cases} y_{12}=\frac{1}{2}x^2,\\ y_{22}=\frac{1}{2}bx^2,\\ y_{32}=\frac{1}{2}x^2,\\ y_{42}=\frac{1}{2}cx^2,\\ y_{52}=\frac{1}{2}x^2,\\ y_{62}=\frac{1}{2}dx^2,\\ y_{72}=\frac{1}{2}x^2,\\ y_{82}=\frac{1}{2}fx^2,\\ y_{92}=e^{-x}+x-1,\\ y_{102}=-2axe^{-x}-2ae^{-x}+2. \end{cases} \quad (45)$$

$$p^3:\begin{cases} y_{13}=\frac{1}{6}bx^3, \\ y_{23}=\frac{1}{6}x^3, \\ y_{33}=\frac{1}{6}cx^3, \\ y_{43}=\frac{1}{6}x^3, \\ y_{53}=\frac{1}{6}dx^3, \\ y_{63}=\frac{1}{6}x^3, \\ y_{73}=\frac{1}{6}fx^3, \\ y_{83}=-e^{-x}+1-x+\frac{1}{2}x^2, \\ y_{93}=2axe^{-x}+4ae^{-x}\,2ax-4a, \\ y_{103}=(1+a^2)(-x^2e^{-x}-2xe^{-x}-2e^{-x}+2). \end{cases} \tag{46}$$

Combining all the terms of (43)-(46) we get

$$y^{(10)}(x)=1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\frac{1}{40320}x^8+\frac{1}{3628800}x^{10}$$
$$+ax+\frac{1}{6}bx^3+\frac{1}{120}cx^5+\frac{1}{5040}dx^7+\frac{1}{362880}fx^9. \tag{47}$$

Using the boundary conditions (39) then we have:

$$a=1.000029332,\quad b=0.9997112299,\quad c=1.002812535,$$
$$d=0.9735681663,\quad f=1.218281800.$$

$$y^{(10)}(x)=1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\frac{1}{40320}x^8+\frac{1}{3628800}x^{10}$$
$$+1.000029332x+0.1666185383x^3+0.0083567711x^5$$
$$+0.0001931682870x^7+0.3357258046E(-5)x^9. \tag{48}$$

The numerical results obtained in Table 1.2. In Table 1.2, we list the results obtained by homotopy perturbation method (HPM) and compared with the exact solution. As we see from this Table, it is clear that the results obtained by the present method are very superior to that obtained by the exact solution highly accurate. As can be seen from Table 1.2, the error decreased when the integer N is decreased.

Table 1.2:

Non-linear tenth-order BVP

X	HPM (N=10)	$y_{exact}=e^x$
0.0	0.100000000E+01	0.100000000E+01
0.2	0.1221408246E+01	0.1221402758E+01
0.4	0.1491833581E+01	0.1491824698E+01
0.6	0.1822127686E+01	0.1822118800E+01
0.8	0.2225546413E+01	0.2225540928E+01
1.0	0.2718281799E+01	0.2718281799E+01

Example 3. Consider the following linear twelve-order problem

$$y^{(12)}(x)=2e^xy^2(x)+y^{(3)}(x),\qquad a<x<b. \tag{49}$$

with the following boundary conditions

$$y^{(2k)}(0)=1,\quad k=0,1,2,3,4,5. \tag{50}$$

$$y^{(2k)}(1)=(1/e),\quad k=0,1,2,3,4,5. \tag{51}$$

The exact solution is

$$y(x)=e^{-x}. \tag{52}$$

Using the transformation (4) we can rewrite the twelve-order boundary value problem (49) as the system of integral equations:

$$\begin{aligned}
y_1 &= 1+\int_0^x y_2(t)dt,\\
y_2 &= A+\int_0^x y_3(t)dt,\\
y_3 &= 1+\int_0^x y_4(t)dt,\\
y_4 &= B+\int_0^x y_5(t)dt,\\
y_5 &= 1+\int_0^x y_6(t)dt,\\
y_6 &= C+\int_0^x y_7(t)dt,\\
y_7 &= 1+\int_0^x y_8(t)dt,\\
y_8 &= D+\int_0^x y_9(t)dt,\\
y_9 &= 1+\int_0^x y_{10}(t)dt,\\
y_{10} &= E+\int_0^x y_{11}(t)dt,\\
y_{11} &= 1+\int_0^x y_{12}(t)dt,\\
y_{12} &= F+\int_0^x [2e^t y^2(t)+y_1^{(3)}(t)dt.
\end{aligned} \tag{53}$$

Comparing the coefficients of like powers of p, we have:

$$p^0:\begin{cases}
y_{10}=1,\\
y_{20}=A,\\
y_{30}=1,\\
y_{40}=B,\\
y_{50}=1,\\
y_{60}=C,\\
y_{70}=1,\\
y_{80}=D,\\
y_{90}=1,\\
y_{100}=E,\\
y_{110}=1,\\
y_{120}=F,
\end{cases} \tag{54}$$

$$p^1:\begin{cases} y_{11}=Ax, \\ y_{21}=x, \\ y_{31}=Bx, \\ y_{41}=x, \\ y_{51}=Cx, \\ y_{61}=x, \\ y_{71}=Dx, \\ y_{81}=x, \\ y_{91}=Ex, \\ y_{101}=x, \\ y_{111}=Fx, \\ y_{121}=2e^{x}-2, \end{cases} \tag{55}$$

$$p^2:\begin{cases} y_{12}=\frac{1}{2}x^2, \\ y_{22}=\frac{1}{2}Bx^2, \\ y_{32}=\frac{1}{2}x^2, \\ y_{42}=\frac{1}{2}Cx^2, \\ y_{52}=\frac{1}{2}x^2, \\ y_{62}=\frac{1}{2}Dx^2, \\ y_{72}=\frac{1}{2}x^2, \\ y_{82}=\frac{1}{2}Ex^2, \\ y_{92}=\frac{1}{2}x^2, \\ y_{102}=\frac{1}{2}Fx^2, \\ y_{112}=2e^{x}-2x-2, \\ y_{102}=4Axe^{x}-4Ae^{x}+4A, \end{cases} \tag{56}$$

Combining all the terms of (54)-(56) we obtain:

$$y^{(12)}(x)=1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\frac{1}{40320}x^8+\frac{1}{3628800}x^{10}+\frac{1}{239500800}x^{12}$$
$$+Ax+\frac{1}{6}Bx^3+\frac{1}{120}Cx^5+\frac{1}{5040}Dx^7+\frac{1}{362880}Ex^9+\frac{1}{39916800}Fx^{11}. \tag{57}$$

Using the boundary conditions (39) then we have:

$A=-0.9999940293,\quad B=-1.000058885,\quad C=-0.9994190942,$

$D=-1.005725028,\quad E=-0.9434337955,\quad F=-1.632120555.$

$$y^{(12)}(x)=1-0.9999940293x+\frac{1}{2}x^2-0.1666764809x^3+\frac{1}{24}x^4-0.008328492451x^5$$
$$+\frac{1}{720}x^6-0.0001995486167x^7+\frac{1}{40320}x^8-0.2599850627E(-5)x^9$$
$$+\frac{1}{3628800}x^{10}-0.4088806105E(-7)x^{11}+\frac{1}{239500800}x^{12}. \tag{58}$$

The numerical results obtained in Table 1.3. In Table 1.3, we list the results obtained by homotopy perturbation method (HPM) and compared with the exact solution. As we see from this Table, it is clear that the results obtained by the present method are very superior to that obtained by the exact solution highly accurate. As can be seen from Table 1.3, the error decreased when the integer N increased.

Table 1.3:

Non-Linear twelfth-order BVP		
X	HPM (N=12)	$y_{exact} = e^{-x}$
0.0	10.00000000E-01	10.00000000E-01
0.2	8.187308703E-01	8.187307531E-01
0.4	6.703208540E-01	6.703200460E-01
0.6	5.488114451E-01	5.488116361E-01
0.8	4.493289646E-01	4.493289641E-01
1.0	3.678794453E-01	3.678794412E-01

4. Conclusion

Homotopy perturbation method is applied to the numerical solution for solving both linear and nonlinear N^{th} boundary value problems. Comparison of the results obtained by the present method with that obtained by exact solution reveals that the present method is very effective and convenient. The numerical results in the Tables [1.1-1.3], show that the present method provides highly accurate numerical results. This method cannot be applied easily if the order of equation is higher than 12^{th} order.

5. References

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