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Existence of Three Weak Solutions for Elliptic Dirichlet Problem

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Abstract
The aim of this paper is to establish the existence of at least three weak solutions for the elliptic Dirichlet problem. Our main tool is a three critical point theorem and Therorem3.1. of Gabriele Bonanno , Giovanni Molica Bisci [4].

Keywords: Dirichlet problem; Critical points; Three noitulos

1. Introduction
In this paper we investigate the following elliptic Dirichlet problem

$$\begin{cases} -\Delta u = \lambda f(x,u) - a(x)u & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \quad (1.1)$$

where Ω is a non empty bounded open subset of the Euclidean space (R^N,|.|), N≥3, with boundary of classC^1, λ is a positive parameter and f: $\Omega \times R \longrightarrow R$ is a function, and the positive weight function a(x)∈ $C(\overline{\Omega})$.
Existence of three solutions for different kinds of Dirichlet problem has been widely studied in literature, see for instance[1, 3, 5, 6,7].

2. Preliminaries
Our main tool is the following critical point theorem.

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Theorem 2.1. Let X be a reflexive real Banach space, $\phi: X \longrightarrow R$ be a coercive, continuously G$\hat{a}$teaux differentiable and sequentially weakly lower semicontinuous functional whose G$\hat{a}$teaux derivative admits a continuous invers on X^*, $\psi: X \longrightarrow R$ be a continuously G$\hat{a}$teaux differentiable functional whose G$\hat{a}$teaux derivative is compact such that $\phi(0) = \psi(0) = 0$.
Assume that there exist r > 0 and $\bar{x} \in X$, with r< $\phi(\bar{x})$, such that:
(a_1) $\frac{sup_{\phi(x)\leq r}\psi(x)}{r} < \frac{\psi(\bar{x})}{\phi(\bar{x})}$;
(a_2) hcae rof$\lambda \in \Lambda_r :=]\frac{\psi(\bar{x})}{\phi(\bar{x})}, \frac{r}{sup_{\phi(x)\leq r}\psi(x)}[$ the functional $\phi - \lambda\psi$ is coercive.
Then, for each $\lambda \in \Lambda_r$, the functional $J_\lambda := \phi - \lambda\psi$ has at least three distinct critical points in X.
Here and in the sequal, f : $\Omega \times R \longrightarrow R$ is a Caratheodory function such that
(h_1) There exist two non negative constants a_1, a_2 and q$\in]1, \frac{2N}{(N-2)}[$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1}, \qquad (2.1)$$

for every (x,t)$\in \Omega \times R$.
We recall that the symbol $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,2}(\Omega)$, with respect to the norm

$$||u||:=(\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}}$$

and we define

$$||u||_I := (\int_\Omega (|\nabla u(x)|^2 + a(x)|u(x)|^2)dx)^{\frac{1}{2}}$$

then there exist positive suitable constants c_1, c_2 such that

$$c_1||u|| \leq ||u||_I \leq c_2\,||u||$$

and a function u:$\Omega \longrightarrow R$ is said to be a weak solution of (1.1) if u$\in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u(x)\nabla v(x)dx - \lambda\int_\Omega f\big(x,u(x)\big)v(x)dx = -\int_\Omega a(x)u(x)v(x)dx$$

for all $v \in H_0^1(\Omega)$.
In order to study problem (1.1), we will use the functionals $\phi, \psi: H_0^1(\Omega) \longrightarrow R$ defined by putting

$$\phi(u) := \frac{||u||_I^2}{2},$$

and

$$\psi(u) := \int_\Omega F\big(x,u(x)\big)dx, \qquad \forall\, u \in H_0^1(\Omega),$$

Where

$$F(x,\xi) := \int_0^\xi f(x,t)dt,$$

for every (x,ξ) $\in \Omega \times R$.
From [4] clearly $\phi: X \longrightarrow R$ is a coercive, continuously G$\hat{a}$teaux differentiable and sequentially weakly lower semi continuous functional whose G$\hat{a}$teaux derivative admits a continuous inverse on X^*. On the other hand, ψ is well defined, continuously G$\hat{a}$teaux differentiable and with compact derivative. More precisely, one has

$$\acute{\phi}(u)(v) = \int_\Omega \big(\nabla u(x).\nabla v(x) + a(x)u(x)v(x)\big)dx,$$

$$\acute{\psi}\,(u)(v) = \int_\Omega f\big(x,u(x)\big)v(x)dx,$$

for every u,v$\in H_0^1(\Omega)$.
A critical point of the functional $J_\lambda := \phi - \lambda\psi$ is a function $u \in H_0^1(\Omega)$ such that

$$\acute{\phi}(u)(v) - \lambda\acute{\psi}(u)(v) = 0, \qquad (2.2)$$

for every v $\in H_0^1(\Omega)$. Hence the critical points of the functional J_λ are weak solutions of problem (1.1). Now, put $2^* = \frac{2N}{(N-2)}$ and denote, as usual, with Γ the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \qquad \forall t > 0.$$

From the Sobolev embedding theorem there exist c$\in R^+$ such that

$$||u||_{L^{2^*}(\Omega)} \leq c||u||, \qquad u \in H_0^1(\Omega). \qquad (2.3)$$

The best constant that appears in (2.3) is

$$c = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma\left(1+\frac{N}{2}\right)}\right)^{\frac{1}{N}}, \qquad (2.4)$$

Fixing q$\in [1, 2^*[$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$||u||_{L^q(\Omega)} \leq c_q||u||, \qquad u \in H_0^1(\Omega), \qquad (2.5)$$

and, in particular, the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact.
Due to (2.4) , as simple consequence of Holder's inequality, it follows that

$$c_q \leq \frac{meas\,(\Omega)^{\frac{2^*-q}{2^*q}}}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma\left(1+\frac{N}{2}\right)}\right)^{\frac{1}{N}}, \qquad (2.6)$$

where meas(Ω) denotes the Lebesgue measure of the set Ω.
Moreover, let

$$D := sup_{x\in\Omega} dist(x, \partial\Omega). \qquad (2.7)$$

Simple calculations show that there is $x_0 \in \Omega$ such that B(x_0,D)$\subseteq \Omega$.
Finally, we set

$$k := \frac{D\sqrt{2}}{2\pi^{\frac{N}{4}}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)}{D^N - \left(\frac{D}{2}\right)^N}\right)^{\frac{1}{2}}, \qquad (2.8)$$

and

$$K_1 := \frac{2\sqrt{2}c_1(2^N-1)}{D^2}, \quad K_2 := \frac{2^{\frac{q+2}{2}}c_q^q(2^N-1)}{qD^2}. \qquad (2.9)$$

3. Conclusion

Our main result is the following theorem.
eroehTm 3.1. Let f: $\Omega \times R \longrightarrow R$ be a Caratheodory function such that (h_1) holds. Assume that
(h_2) $F(x,\xi) \geq 0$ for every (x,ξ) $\in \Omega \times R^+$;
(h_3) there exist two positive constants b and s < 2 such that

$$F(x,\xi) \leq b(1 + |\xi|^s),$$

for almost every x$\in \Omega$ and for every $\xi \in R$;
(h_4) there exist two positive constants γ and δ, with $\delta > \gamma k$ such that

$$\frac{inf_{\,x\in\Omega}F(x,\delta)}{E\delta^2} < a_1\frac{K_1}{\gamma} + a_2K_2\gamma^{q-2},$$

where a_1, a_2 are given in (h_1) and k, K_1, K_2 are given by (2.8) and (2.9).
Then, for each parameter λ belonging to

$$\Lambda_{(\gamma,\delta)} :=] \frac{2(2^N-1)}{D^2}\frac{\delta^2 E}{\inf_{x\in\Omega}F(x,\delta)}, \frac{2(2^N-1)}{D^2}\frac{1}{\left(a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}\right)}[,$$

the problem (1.1) possesses at least three weak solutions in $H_0^1(\Omega)$.
Proof: Let us apply theorem 2.1 with X= $H_0^1(\Omega)$ and

$$\phi(u) := \frac{||u||_l^2}{2}, \quad \psi(u) := \int_\Omega F(x,u(x))dx,$$

for every u∈ X. Let λ>0 and put

$$J_\lambda(u) := \phi(u) - \lambda\psi(u), \qquad \forall\, u \in X.$$

As observed in section 2, $\phi : X \longrightarrow R$ is a coercive, continuously G$\hat{a}$teaux differentiable and sequentially weakly lower semicontinuous functional whose G$\hat{a}$teaux derivative admits a continuous inverse on X^*. Moreover, ψ is continuously G$\hat{a}$teaux differentiable with compact derivative and $\phi(0) = \psi(0) = 0$.

Owing to (h_1), one has that

$$F(x,\xi) \le a_1|\xi| + a_2\frac{|\xi|^q}{q}, \qquad (3.1)$$

for every $(x,\xi) \in \Omega \times R$.
Let r∈$]0,+\infty[$ and consider the function

$$\chi(r) := \frac{\sup_{u\in\phi^{-1}(]-\infty,r])}\psi(u)}{r}.$$

Taking into account (3.1) it follows that

$$\psi(u) = \int_\Omega F(x,u(x))dx \le a_1||u||_{L^1(\Omega)} + \frac{a_2}{q}||u||_{L^q(\Omega)}^q.$$

Then, for every u∈ X: $\phi(u) \le r$, due to (2.5), we get

$$\psi(u) \le (\sqrt{2r}c_1a_1 + \frac{2^{\frac{q}{2}}c_q^q a_2}{q}\, r^{\frac{q}{2}}).$$

Hence

$$sup_{u\in\phi^{-1}(]-\infty,r])}\psi(u) \le (\sqrt{2r}c_1a_1 + \frac{2^{\frac{q}{2}}c_q^q a_2}{q}\, r^{\frac{q}{2}}). \qquad (3.2)$$

Since, from (3.2), the following inequality holds

$$\chi(r) \le \left(\sqrt{\frac{2}{r}}c_1a_1 + \frac{2^{\frac{q}{2}}c_q^q a_2}{q}\, r^{\frac{q}{2}-1}\right), \qquad (3.3)$$

for every r>0.
Next, put

$$u_\delta(x) := \begin{cases} 0 & if\ x \in \Omega - B(x_0,D), \\ \frac{2\delta}{D}(D-|x-x_0|) & if\ x \in B(x_0,D) - B\left(x_0,\frac{D}{2}\right), \\ \delta & if\ x \in B\left(x_0,\frac{D}{2}\right). \end{cases} \qquad (3.4)$$

Clearly $u_\delta \in X$and we have

$$\phi(u_\delta) = \frac{1}{2}(\int_\Omega (|\nabla u_\delta(x)|^2 + a(x)|u_\delta(x)|^2)dx$$

$=\frac{1}{2}(\int_{B(x_0,D)-B\left(x_0,\frac{D}{2}\right)}\frac{(2\delta)^2}{D^2}dx+\int_{B(x_0,D)-B\left(x_0,\frac{D}{2}\right)}a(x)\frac{(2\delta)^2}{D^2}\left|D-|x-x_0|\right|^2dx+$
$\int_{B(x_0,D)}a(x)\delta^2dx)$

$$\leq\frac{1}{2}\left(\frac{(2\delta)^2}{D^2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(D^N-\left(\frac{D}{2}\right)^N\right)\right.$$

$$+\frac{(2\delta)^2}{D^2}sup_{x\in\Omega}\mathrm{a(x)}\,max_{x\in B(x_0,D)-B\left(x_0,\frac{D}{2}\right)}\left|D-|x-x_0|\right|^2\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(D^N-\left(\frac{D}{2}\right)^N\right)$$

$$\left.+\delta^2 sup_{x\in\Omega}\mathrm{a(x)}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{D^N}{2^N}\right)$$

$$=\frac{1}{2}\frac{(2\delta)^2}{D^2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(D^N-\left(\frac{D}{2}\right)^N\right)\left(1+Mh+\frac{D^2}{2^2}M\frac{1}{2^N-1}\right)$$

$$=\frac{E}{2}\frac{(2\delta)^2}{D^2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(D^N-\left(\frac{D}{2}\right)^N\right). \qquad (3.5)$$

Bearing in mind that $\delta>\gamma k$, it follows that $\gamma^2<\phi(u_\delta)$.
At this point, by (h_2) we infer

$$\psi(u_\delta)=\int_\Omega F\left(x,u_\delta(x)\right)dx\geq\int_{B\left(x_0,\frac{D}{2}\right)}F(x,\delta)dx\geq$$
$$inf_{x\in\Omega}F(x,\delta)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{D^N}{2^N}. \qquad (3.6)$$

Hence, by (3.5) and (3.6) one has

$$\frac{\psi(u_\delta)}{\phi(u_\delta)}\geq\frac{D^2}{2(2^N-1)}\frac{inf_{x\in\Omega}F(x,\delta)}{\delta^2E}. \qquad (3.7)$$

In view of (3.3) and taking into account (h_4), we get

$$\chi(\gamma^2)=\frac{sup_{u\in\phi^{-1}(]-\infty,\gamma^2[)}\psi(u)}{\gamma^2}\leq\left(\sqrt{2}\frac{c_1}{\gamma}a_1+\frac{2^{\frac{q}{2}}c_q^q a_2}{q}\gamma^{q-2}\right)$$
$$=\frac{D^2}{2(2^N-1)}\left(a_1\frac{K_1}{\gamma}+a_2K_2\gamma^{q-2}\right)<\frac{D^2}{2(2^N-1)}\frac{inf_{x\in\Omega}F(x,\delta)}{\delta^2E}\leq\frac{\psi(u_\delta)}{\phi(u_\delta)}.$$

Therefore, the assumption (a_1) of theorem 2.1 is satisfied.

Moreover, if s <2, for every u$\in X$, $|u|^s\in L^{\frac{2}{s}}(\Omega)$ and the Holder's inequality gives

$$\int_\Omega|u(x)|^sdx\leq||u||_{L^2(\Omega)}^s\,meas(\Omega)^{\frac{2-s}{2}},\qquad\forall u\in X.$$

Then, by (2.5), one has

$$\int_\Omega|u(x)|^sdx\leq c_2^s||u||^s\,meas(\Omega)^{\frac{2-s}{2}},\qquad\forall u\in X. \qquad (3.8)$$

From (3.8) and due to condition (h_3), it follows that

$$J_\lambda(u)\geq\frac{\left||u|\right|_I^2}{2}-\lambda b meas(\Omega)^{\frac{2-s}{2}}||u||^s-\lambda b\,meas(\Omega),\qquad\forall u\in X.$$

Therefore, J_λ is a coercive functional for every positive parameter, in particular, for every

$$\lambda \in \Lambda_{(\gamma,\delta)} \subseteq]\frac{\phi(u_\delta)}{\psi(u_\delta)}, \frac{\gamma^2}{sup_{\phi(u)\leq\gamma^2}}[.$$

Then, also condition (a_2) holds. Hence, all the assumptions of theorem 2.1 are satisfied, so that, for each $\lambda \in \Lambda_{(\gamma,\delta)}$ the functional J_λ has at least three distinct critical points that are weak solutions of the problem (1.1).

Example 3.1 Let Ω be an open ball of radius one in R^4, q:=3∈]2,4[and s:=$\frac{3}{2} < 2$.

Pick r:=200 and consider the function f : R ⟶ R defined by

$$f(t):=\begin{cases} 1+t^2 & if\ t \leq 200, \\ 1+2000\sqrt{2t} & if\ t > 200. \end{cases} \qquad (3.9)$$

and a(x) = $\frac{1}{e^2 x^{\frac{1}{3}}}$.

Then, by theorem 3.1, for each

$$\lambda \in]\frac{18000\,E}{40003}, \frac{12^{\frac{1}{4}}}{1+2\sqrt{3}\pi^2} 4\pi[,$$

the problem(1.1) possesses at least three weak positive solutions in $H_0^1(\Omega)$.

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