The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

Journal of Mathematics and Computer Science Vol .1 No.1 (2010) 28-32

On W_2 –Curvature Tensor N(k)-Quasi Einstein Manifolds

A. Taleshian[1,*] **and A. A. Hosseinzadeh**

[1]*Department of Mathematics, Faculty of Sciences,*
Mazandaran University.P.O.Box 47416-1467,Babolsar, Iran.

* *taleshian@umz.ac.ir*

Received: Sep 2009, Revised: Nov 2009
Online Publication: Apr 2010

ABSTRACT

We consider N(k)-quasi Einstein manifolds satisfying the conditions $R(\xi,X).W_2 = 0$, $W_2(\xi,X).S = 0$, $P(\xi,X).W_2 = 0$, **where** W_2 **and** P **denote the** W_2 **-curvature tensor and projective curvature tensor, respectively.**

M.S.C: 53C25.

Keywords: *k*-nullity distribution - quasi Einstein manifold - *N(k)*-quasi Einstein manifold - W_2 -curvature tensor - projective curvature tensor.

1. Introduction

The notion of a quasi Einstein manifold was introduced by M. C. Chaki in [1]. A non flat n-dimensional Riemannian manifold (M,g) is said to be a quasi Einstein manifold if its Ricci tensor S satisfies

$$(1.1) \qquad S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \qquad \forall\ X,Y \in TM$$

for some smooth functions a and $b \neq 0$, where η is a non zero 1-forms such that

$$(1.2) \qquad g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = \eta(\xi) = 1,$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If $b \neq 0$ then the manifold reduced to an Einstein manifold. For more details about quasi Einstein manifolds see also ([2]-[6]).
If the generator ξ belongs to k-nullity distribution *N(k)* then the quasi Einstein manifold is called as an *N(k)*-quasi Einstein manifold [12]. In [12], it was shown that a conformally flat quasi Einstein manifold is an *N(k)*-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an *N(k)*-quasi Einstein manifold. The derivation conditions $R(\xi,X).R = 0$ and $R(\xi,X).S = 0$ were also studied in [12], where R and S denote the curvature and Ricci

tensor, respectively. In [9], it was proved that in an n-dimensional N(k)-quasi Einstein manifold $k = \frac{a+b}{n-1}$. In [7], derivation conditions $R(\xi,X).\rho = 0$, $\rho(\xi,X).S = 0$ and $\rho(\xi,X).\rho = 0$ were studied where ρ is the projective curvature tensor, also it physical examples of N(k)-quasi Einstein manifolds were given. The derivation conditions $R(\xi,X).C = 0$, $R(\xi,X).\tilde{C} = 0$, studied in [8], where C and $\tilde{C}$ denote the conformal curvature tensor and quasi conformal curvature tensor, respectively. In this paper, we consider *N(k)*-quasi Einstein manifolds satisfying the conditions $R(\xi,X).W_2 = 0$, $W_2(\xi,X).S = 0$, $P(\xi,X).W_2 = 0$, where W_2 and P denote the W_2-curvature tensor and the projective curvature tensor, respectively.

2. N(k)-quasi Einstein manifolds

From (1.1) and (1.2) we obtain

$$S(X,\xi) = (a+b)\eta(X) \tag{2.1}$$

$$r = na + b, \tag{2.2}$$

where r is the scalar curvature of M.
The Ricci operator Q of a Riemannian manifold (M,g) is defined by

$$S(X,Y) = g(QX,Y).$$

If (M,g) is a quasi Einstein manifold [1], its Ricci operator satisfies

$$Q = aI + b\eta \otimes \xi. \tag{2.3}$$

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [11], of a Riemannian manifold defined by

$$N(k): p \to N_p(k) = \{Z \in T_pM \mid R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}\}$$

for all $X, Y \in TM^n$, where k is some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then is said to be an *N(k)*-quasi Einstein manifold [12].
Lemma2.1. [9] In an n-dimensional *N(k)*-quasi Einstein manifold it follows that

$$k = \frac{a+b}{n-1}. \tag{2.4}$$

Let (M^n, g) be an *N(k)*-quasi Einstein manifold. Then, we have [12]

$$R(Y,Z)\xi = \frac{a+b}{n-1}\{\eta(Z)Y - \eta(Y)Z\}. \tag{2.5}$$

The equation (2.5) is equivalent to

$$R(\xi,Y)Z = \frac{a+b}{n-1}\{g(Y,Z)\xi - \eta(Z)Y\} = -R(Y,\xi)Z. \tag{2.6}$$

In [7], we view the following physical examples of *N(k)*-quasi Einstein manifolds.
Example2.2. [7] A conformally flat perfect fluid space-time (M^4, g) satisfying Einstein's equation without cosmological constant is an $N\left(\frac{k(3\sigma + p)}{6}\right)$-quasi Einstein manifold.
Example2.3. [7] A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation with cosmological constant is an $N\left(\frac{\lambda}{3} + \frac{k(3\sigma + p)}{6}\right)$-quasi Einstein manifold.
where σ is the energy density and p is the isotropic pressure of the fluid.

3. The W_2-curvature tensor of an N(k)-quasi Einstein manifold

In [10], Pokhariyal and Mishra have defined a new curvature tensor

$$(3.1) \qquad W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\left[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)\right],$$

where *S* is a Ricci tensor of type (0,2).

Proposition 3.1. In an n-dimensional *N(k)*-quasi Einstein manifold *M*, the W_2-curvature tensor satisfies

$$(3.2) \qquad \eta(W_2(X,Y)Z) = 0,$$

for all vector fields *X*, *Y* , *Z* on *M*.

Proof. From (1.1), (2.5), (2.6) and (3.1) the Eq. (3.2) follow easily. □

Theorem 3.2. Let *M* be an *N(k)*-quasi Einstein manifold. Then M satisfies the condition R(ξ,X).W_2 = 0 if and only if a + b = 0 or W_2 = 0.

Proof. Assume that M is an n-dimensional N(k)-quasi Einstein manifold satisfying the condition *R(ξ,X).*W_2 =0. Since M satisfies the condition *R(ξ,X).*W_2 = 0 we can write

$$0 = R(\xi, X)W_2(Y,Z)U - W_2(R(\xi, X)Y, Z)U$$
$$- W_2(Y, R(\xi, X)Z)U - W_2(Y,Z)R(\xi, X)U$$

for all vector fields *X, Y, Z, U* on *M*.
Using (2.6), in above equation we get

$$0 = \frac{a+b}{n-1}\{W_2(Y,Z,U,X)\xi - \eta(W_2(Y,Z)U)X$$
$$- g(X,Y)W_2(\xi,Z)U + \eta(Y)W_2(X,Z)U$$
$$- g(X,Z)W_2(Y,\xi)U + \eta(Z)W_2(Y,X)U$$
$$- g(X,U)W_2(Y,Z)\xi + \eta(U)W_2(Y,Z)X\}$$

which implies either a + b = 0 or

$$0 = W_2(Y,Z,U,X)\xi - \eta(W_2(Y,Z)U)X$$
$$- g(X,Y)W_2(\xi,Z)U + \eta(Y)W_2(X,Z)U$$
$$- g(X,Z)W_2(Y,\xi)U + \eta(Z)W_2(Y,X)U$$
$$- g(X,U)W_2(Y,Z)\xi + \eta(U)W_2(Y,Z)X.$$

Taking the inner product of above equation with ξ we obtain

$$(3.3) \qquad 0 = W_2(Y,Z,U,X) - \eta(W_2(Y,Z)U)\eta(X)$$
$$- g(X,Y)\eta(W_2(\xi,Z)U) + \eta(Y)\eta(W_2(X,Z)U)$$
$$- g(X,Z)\eta(W_2(Y,\xi)U) + \eta(Z)\eta(W_2(Y,X)U)$$
$$- g(X,U)\eta(W_2(Y,Z)\xi) + \eta(U)\eta(W_2(Y,Z)X).$$

Using (3.2) into (3.3) we obtain

$$W_2(Y,Z,W,X) = 0.$$

The converse statement is trivial. This completes the proof of the theorem. □

Next, we have the following theorem

Theorem 3.3. Let *M* be an *N(k)*-quasi Einstein manifold. Then *M* satisfies the condition $W_2(\xi,X).S = 0$ if and only if $a = 0$.

Proof. Since $W_2(\xi,X).S = 0$, we have

$$0 = S(W_2(\xi,X)Y,Z) + S(Y,W_2(\xi,X)Z) . \tag{3.4}$$

In view of (1.1), (3.1) and (3.2) in (3.4) we have

$$0 = \frac{ab}{n-1}\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\mu(Z)\}. \tag{3.5}$$

From (3.5) by a contraction, we obtain

$$ab = 0. \tag{3.6}$$

Since $b \neq 0$, then from (3.6) we have $a = 0$. The converse statement is trivial. This completes the proof of the theorem. □

Let (M^n, g) be a Riemannian manifold. The projective curvature tensor [13], is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{3.7}$$

If *P* be a projective curvature tensor in an n-dimensional *N(k)*-quasi Einstein manifold, we have [7]

$$P(\xi,X)Y = \frac{b}{n-1}[g(X,Y)\xi - \eta(X)\eta(Y)\xi]. \tag{3.8}$$

Theorem 3.4. Let *M* be an *N(k)*-quasi Einstein manifold. Then M satisfies the condition $P(\xi,X).W_2 = 0$ if and only if $W_2 = 0$.

Proof. Assume that *M* is an n-dimensional *N(k)*-quasi Einstein manifold satisfying the condition $P(\xi,X).W_2 = 0$. Since M satisfies the condition $P(\xi,X).W_2 = 0$ we can write

$$0 = P(\xi,X)W_2(Y,Z)U - W_2(P(\xi,X)Y,Z)U \tag{3.9}$$

$$- W_2(Y,P(\xi,X)Z)U - W_2(Y,Z)P(\xi,X)U$$

for all vector fields *X, Y, Z, U* on *M*.
Using (3.8), in (3.9) equation we get

$$\begin{aligned} 0 = \frac{b}{n-1}\{ & W_2(Y,Z,U,X)\xi - \eta(W_2(Y,Z)U)\eta(X)\xi \\ & - g(X,Y)W_2(\xi,Z)U + \eta(X)\eta(Y)W_2(\xi,Z)U \\ & - g(X,Z)W_2(Y,\xi)U + \eta(Z)\eta(X)W_2(Y,\xi)U \\ & - g(X,U)W_2(Y,Z)\xi + \eta(X)\eta(U)W_2(Y,Z)\xi\}. \end{aligned}$$

Since $b \neq 0$, then

$$(3.10) \qquad 0 = W_2(Y,Z,U,X)\xi - \eta(W_2(Y,Z)U)\eta(X)\xi$$
$$- g(X,Y)W_2(\xi,Z)U + \eta(X)\eta(Y)W_2(\xi,Z)U$$
$$- g(X,Z)W_2(Y,\xi)U + \eta(Z)\eta(X)W_2(Y,\xi)U$$
$$- g(X,U)W_2(Y,Z)\xi + \eta(X)\eta(U)W_2(Y,Z)\xi.$$

Taking the inner product of (3.10) with ξ we obtain

$$(3.11) \qquad 0 = W_2(Y,Z,U,X) - \eta(W_2(Y,Z)U)\eta(X)$$
$$- g(X,Y)\eta(W_2(\xi,Z)U) + \eta(X)\eta(Y)\eta(W_2(\xi,Z)U)$$
$$- g(X,Z)\eta(W_2(Y,\xi)U) + \eta(Z)\eta(X)\eta(W_2(Y,\xi)U)$$
$$- g(X,U)\eta(W_2(Y,Z)\xi) + \eta(X)\eta(U)\eta(W_2(Y,Z)\xi).$$

Using (3.2) into (3.3) we obtain

$$W_2(Y,Z,W,X) = 0.$$

The converse statement is trivial. This completes the proof of the theorem. □

References

[1] M. C. Chaki, R. K. Maity, On quasi Einstein manifolds, Publ. Math. Debrecen 57 (2000), 297-306.

[2] M. C. Chaki, P. K. Ghoshal, Some global properties of quasi Einstein manifolds. Publ. Math. Debrecen 63(4) (2003), 635-641.

[3] U. C. De, G. C. Ghosh, On quasi Einstein manifolds. Period Math Hungar 48(1-2) (2004), 223-231.

[4] U. C. De, G. C. Ghosh, On quasi Einstein manifolds. II. Bull Calcutta Math Soc 96(2) (2004), 135-138.

[5] U. C. De, G. C. Ghosh, On conformally flat special quasi Einstein manifolds. Publ Math Debrecen 66(1-2) (2005), 129-136.

[6] U. C. De, J. Sengupta, D. Saha, Conformally flat quasi-Einstein spaces. Kyungpook Math J 46(3) (2006):417-423.

[7] C. Özgür, N(k)-quasi Einstein manifolds satisfying certain conditions, Chaos, Solitons and Fractals 38 (2008) 1373-1377.

[8] C. Özgür, Sibel Sular, On N(k)-quasi Einstein manifolds satisfying certain conditions, Balkan Journal of Geometry and Its Applications, Vol.13 (2008), 74-79.

[9] C. Özgür, M. M. Tripathi On the concircular curvature tensor of an N(k)-quasi Einsteinmanifold, Math. Pannon. 18 (2007), 95-100.

[10] G.P. Pokhariyal and R.S. Mishra,, The curvature tensor and their relativistic significance. Yokohoma Math. J. 18 (1970), 105–108.

[11] S. Tanno, Ricci curvatures of contact Riemannian manifolds, Tˆohoko Math. J., 40 (1988), 441-448.

[12] M. M. Tripathi, J. S. Kim, On N(k)-quasi Einstein manifolds, Commun. Korean Math. Soc. 22 (2007) 411-417.

[13] Yano K, Kon M., Structures on manifolds. Series in pure mathematics, vol. 3. Singapore: World Scienti.c Publishing Co.; (1984)