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Quasilinearization and Numerical Solution of Nonlinear Volterra Integro-differential Equations

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Abstract
When we use the projection methods in order to obtain the approximation solution of nonlinear equations, we always have some difficulties such as solving nonlinear algebraic systems. The method of generalized quasilinearization when is applied to the nonlinear integro-differential equations of Volterra type, gives two sequences of linear integro-differential equations with solutions monotonically and quadratically convergent to the solution of nonlinear equation. In this paper we employ step-by-step collocation method to solve the linear equations numerically and then approximate the solution of the nonlinear equation. In this manner we do not encounter solving nonlinear algebraic systems. Error analysis of the method is performed and to show the accuracy of the method some numerical examples are proposed.

Keywords: Volterra integro-differential equation; Collocation method; Quasilinearization technique.

2010 Mathematics Subject Classification: Primary 54A40; Secondary 46S40.

1. Introduction
It is well known that the method of quasilinearization [1] provides an excellent tool for obtaining approximated solutions of nonlinear differential equations. This technique works fruitfully for the problems that their nonlinear parts involve convex or concave functions and gives two sequences of linear problems that their solutions are upper and lower solutions to the nonlinear problem and are converging monotonically and quadratically to the unique solution of the given nonlinear problem. Recently, this method is applied to a variety of problems [2-5] and in the continuation the convexity assumption was relaxed and the method was generalized and extended in various directions to make it applicable to a large class of problems [6-15].

The method of quasilinearization is also an effective tool to obtain lower or upper bounds for the solutions of nonlinear differential equations [1,17]. To describe it, consider the initial-value problem (IVP)

$$x'(t)=f\left(t,x\left(t\right)\right),\qquad x\left(0\right)=x_0, \tag{1}$$

on $J=[0,T]$. If f is convex on x, then we can find a function $g(t,x,y)$ which is linear in x such that

$$f(t,x)=\max_y g(t,x,y).$$

By choosing an initial approximation $y_0(t)$, using $g(t,x,y)$ one can generate a monotone sequence $y_n(t)$, that converges quadratically to the unique solution of Eq. (1). Moreover, the sequence provides good lower bounds for the solution. On the other hand, if f is concave on x, same results hold that offer monotone approximations along with similar properties and good upper bounds.

Consider the IVP

$$x'(t)=f\left(t,x\left(t\right)\right)+\int_{t_0}^{t}k\left(t,s,x\left(s\right)\right)ds,\qquad x\left(t_0\right)=x_0, \tag{2}$$

With

$$t\in J=\left[t_0,t_0+T\right],\quad t_0\geq 0,\quad T>0.$$

When the numerical methods are applied to solve these nonlinear problems, they mostly are converted to a nonlinear algebraic system. Eq. (2) is studied in [17] using multistep rules with quadrature formulas and in [18] using collocation method and in both of them the integral terms are discretized to nonlinear algebraic systems. But these nonlinear systems need some conditions to have a unique solution and require an iteration method (In many cases the Newton's iteration) and a suitable starting point to be convergent to the solution. When we use the multistep methods, the process of solving these nonlinear algebraic systems is repeated in each step size to obtain the next step nodes and this process causes a lot of computational costs and additional works. In equation when f and k are convex in x, the method of quasilinearization [11] is applicable and offers the following two linear iterative schemes

$$\begin{aligned}\alpha'_{p+1}(t)&=f\left(t,\alpha_p\left(t\right)\right)+f_x\left(t,\alpha_p\left(t\right)\right)\left(\alpha_{p+1}(t)-\alpha_p\left(t\right)\right)\\&\quad+\int_{t_0}^{t}\left[k\left(t,s,\alpha_p\left(s\right)\right)+k_x\left(t,s,\alpha_p\left(s\right)\right)\left(\alpha_{p+1}(t)-\alpha_p\left(t\right)\right)\right]ds,\qquad \alpha_{p+1}\left(t_0\right)=x_0,\end{aligned} \tag{3}$$

$$\begin{aligned}\beta'_{p+1}(t)&=f\left(t,\beta_p\left(t\right)\right)+f_x\left(t,\alpha_p\left(t\right)\right)\left(\beta_{p+1}(t)-\beta_p\left(t\right)\right)\\&\quad+\int_{t_0}^{t}\left[k\left(t,s,\beta_p\left(s\right)\right)+k_x\left(t,s,\alpha_p\left(s\right)\right)\left(\beta_{p+1}(t)-\beta_p\left(t\right)\right)\right]ds,\qquad \beta_{p+1}\left(t_0\right)=x_0,\end{aligned} \tag{4}$$

for $p=0,1,2,\ldots$, as two linear integro-differential equations, where $\alpha_0(t)$ and $\beta_0(t)$ are the upper and lower solutions of Eq. (2), presented in definition 1. The solutions of the iterative schemes (3) and (4) are quadratically and monotonically convergent to the unique solution of Eq. (2). In this paper we apply step-by-step collocation method in a piecewise continuous polynomials space to solve the linear equations (3) numerically. We combine this method and the iterative schemes (3) (where with respect to their linearity and quadratically convergent is rapid in convergence) to approximate the unique solution of Eq. (2).

This paper has been organized as follows: A general framework of the idea of quasilinearization used to solve the nonlinear integro-differential equations and some conclusions are recalled in section 2. Section 3 shows employing step-by-step collocation method in approximating the solution of the linear integro-differential equations in a piecewise continuous polynomials space.

Section 4 includes discretization to a linear algebraic system and discussion of the convergence of the method and in section 5, the suggested method is applied on some numerical examples.

2. Integro-differential inequalities and quasilinearization

Consider the nonlinear integro-differential equation

$$x'(t)=f\left(t,x(t)\right)+\int_0^t k\left(t,s,x(s)\right)ds, \qquad x(0)=x_0, \tag{5}$$

For $f\in C\left[J\times\mathbf{R},\mathbf{R}\right]$ and $k\in C\left[\mathrm{D}\times\mathbb{R},\mathbb{R}\right]$ where $T\in\mathbb{R}$ and $t\in J\left[0,T\right]$ and $\mathrm{D}=\left\{(t,s)\in J\times J: s\le t\right\}$. Eq. (2) is reducible to Eq. (5) by substituting $t-t_0$ instead of t.

Definition 1. if a function $\alpha\in C^1\left[J,\mathbb{R}\right]$ satisfies the inequality

$$\alpha'(t)\le f\left(t,\alpha(t)\right)+\int_0^t k\left(t,s,\alpha(s)\right)ds, \qquad \alpha(0)\le x_0,$$

then α is said to be a lower solution of Eq. (5) on J and an upper solution If the reversed inequality is satisfied.

We have the following conclusions about lower and upper solutions of the IVP (5).

Lemma 1. ([11]): Consider the IVP (5) and that:

(**A**1) $f\in C\left[J\times\mathbb{R},\mathbb{R}\right]$ and $k\in C\left[\mathrm{D}\times\mathbb{R},\mathbb{R}\right]$ and $k\left(t,s,x\right)$ is monotone nondecreasing in x for each fixed $(t,s)\in\mathrm{D}$;

(**A**2) $\alpha_0(t),\beta_0(t)\in C^1\left[J,\mathbb{R}\right]$ are lower and upper solutions of the IVP (5) respectively;

(**A**3) for $(t,s)\in\mathrm{D}, v\ge w, L\ge 0, f\left(t,v\right)-f\left(t,w\right)\le L\left(v-w\right)$, and $k\left(t,s,v\right)-k\left(t,s,w\right)\le N\left(v-w\right), N>0$.

Then we have $\alpha_0(t)\le\beta_0(t)$ for $t\in J$, provided $\alpha_0(t_0)\le\beta_0(t_0)$.

If the Lemma 1 holds, it is shown that the IVP (5) has a unique solution $x(t)$ such that satisfies in the relation

$$\alpha_0(t)\le x(t)\le\beta_0(t), \qquad t\in J.$$

Using the norm $\|x\|=\max_{t\in J}|x(t)|$ and defining two iterative schemes (3) and (4), the following theorem is applied for the unique solution of (5):

Theorem 1. ([11]): Assume that:

(**B**1) $f\in C^2\left[J\times\mathbb{R},\mathbb{R}\right]$, $k\in C^2\left[\mathrm{D}\times\mathbb{R},\mathbb{R}\right]$ and $\alpha_0(t),\beta_0(t)\in C^1\left[J,\mathbb{R}\right]$ are lower and upper solutions of the IVP (5) such that $\alpha_0(t)\le\beta_0(t)$ on J;

(**B**2) $f_{xx}\left(t,x\right)\ge 0$ for each $t\in J$ and $k_{xx}\left(t,s,x\right)\ge 0$ for each $(t,s)\in\mathrm{D}$;

(**B**3) $k\left(t,s,x\right)$ is monotone nondecreasing in x for each $(t,s)\in\mathrm{D}$ and for each $\alpha_0(t)\le v(t)\le\beta_0(t)$

Then the monotone sequences $\left\{\alpha_p(t)\right\}$ and $\left\{\beta_p(t)\right\}$ generated by iterative schemes (3) and (4) converge uniformly and quadratically to the unique solution of (5) on J,

$$\left\|x-\alpha_{p+1}\right\|\le\frac{2e^{LT}}{\sqrt{L^2+4L_1}}\left(M+M_1T\right)\left\|x-\alpha_p\right\|^2,$$

$$\left\|\beta_{p+1}-x\right\| \le \frac{2e^{LT}}{\sqrt{L^2+4L_1}}\left(M+M_1T\right)\left\|\beta_p-x\right\|^2,$$

$$L=\max_{t\in J}\left|f_x\left(t,x\right)\right|, \qquad L_1=\max_{t\in J}\left|k_x\left(t,s,x\right)\right|,$$
$$M=\max_{t\in J}f_{xx}\left(t,x\right), \qquad M_1=\max_{t\in J}k_{xx}\left(t,s,x\right),$$

And satisfy the relation

$$\alpha_0(t)\le\alpha_1(t)\le\cdots\le\alpha_p(t)\le\beta_p(t)\le\cdots\le\beta_1(t)\le\beta_0(t).$$

The following Lemma in [19] is required to establish the convergence of the presented method.

Lemma 2. Suppose $\|.\|$ is a subordinate matrix norm for which the norm of the identity matrix $\|I\|=1$ and E is a matrix such that $\|E\|<1$.Then $(I-E)$ is nonsingular and

$$\left\|(I-E)^{-1}\right\|\le\left(1-\|E\|\right)^{-1}.$$

3. Piecewise Polynomials Collocation Method

Consider the partition $\{0=t_0<t_1<\cdots<t_N=T\}$ on J, $h_n=(t_{n+1}-t_n):n=0,\dots,N-1$, $h=\max_n\{h_n\}$, and indicate the above partition by J_h. According to this partition we have the following definition.

Definition 2. Suppose that J_h is a given partition on J. The piecewise polynomials space $S_m^{(d)}(J_h)$ with $m\ge 0$, $-1\le d\le m$ is defined by

$$S_m^{(d)}(J_h)=\left\{q(t)\in C^d[J,\mathbb{R}]:q\big|_{\sigma_n}\in\pi_m;0\le n\le N-1\right\}.$$

Here $\sigma_n=(t_n,t_{n+1}]$ and π_m denotes the space of polynomials of degree not exceeding m. Now the linear integro-differential equation (3) may be re-written in the form of

$$\alpha_p'(t)=H_p(t)+f_p(t)\alpha_p(t)+\int_0^t k_p(t,s)\alpha_p(s)ds, \qquad \alpha_p(0)=x_0, \tag{6}$$

where

$$\begin{aligned}H_p(t)=&f\left(t,\alpha_{p-1}(t)\right)-f_x\left(t,\alpha_{p-1}(t)\right)\alpha_{p-1}(t)\\&+\int_0^t\left(k\left(t,s,\alpha_{p-1}(s)\right)-k_x\left(t,s,\alpha_{p-1}(s)\right)\alpha_{p-1}(s)\right)ds,\end{aligned} \tag{7}$$

and

$$f_p(t)=f_x\left(t,\alpha_{p-1}(t)\right), \qquad k_p(t,s)=k_x\left(t,s,\alpha_{p-1}(s)\right). \tag{8}$$

We approximate the solution of the linear IVP (3) in the continuous polynomials space

$$S_m^{(0)}(J_h)=\left\{q(t)\in C[J,\mathbb{R}]:q\big|_{\sigma_n}\in\pi_{m-1};0\le n\le N-1\right\},$$

By collocation method corresponding to the choice $d=0$ in Definition 2 . The collocation solution is denoted by $\hat{\alpha}_p(t)$ and defined by the collocation equation

$$\hat{\alpha}'_p(t) = H_p(t) + f_p(t)\hat{\alpha}_p(t) + \int_0^t k_p(t,s)\hat{\alpha}_p(s)ds, \qquad \hat{\alpha}_p(0) = x_0, \quad t \in X_h, \tag{9}$$

where X_h contains the collocation points

$$X_h = \{t_n + c_i h_n : 0 \le c_1 \le \cdots \le c_m \le 1; 0 \le n \le N-1\}, \tag{10}$$

and is determined by the points of the partition J_h and the given collocation parameters $\{c_i\} \in [0,1]$.

4. Lagrange Basis Functions and Discretization

When the Lagrange polynomials are used as basic functions in each subinterval σ_n for the space $S_m^{(0)}(J_h)$, the collocation equation has a simple form. Lagrange polynomials in σ_n can be written as

$$L_j(z) = \prod_{k \ne j}^{m} \frac{z - c_k}{c_j - c_k}, \qquad z \in [0,1], \quad j = 1,\ldots,m. \tag{11}$$

where belong to π_{m-1}. Also set

$$A_{n,j}^p = \hat{\alpha}'_p(t_n + c_j h_n), \qquad j = 1,\ldots,m. \tag{12}$$

The restriction of the collocation solution $\hat{\alpha}_p(t) \in S_m^{(0)}(J_h)$ to the subinterval σ_n is as

$$\hat{\alpha}'_{p,n}(z) = \hat{\alpha}'_p(t_n + zh_n) = \sum_{j=1}^{m} L_j(z) A_{n,j}^p, \qquad z \in (0,1] \tag{13}$$

and by letting $\mathsf{L}_j(z) = \int_0^z L_j(v)dv$ and $y_n = \hat{\alpha}_p(t_n)$ the representation of $\hat{\alpha}_p(t)$ on σ_n is obtained as follows:

$$\hat{\alpha}_{p,n}(z) = \hat{\alpha}_p(t_n + zh_n) = y_n + h_n \sum_{j=1}^{m} \mathsf{L}_j(z) A_{n,j}^p, \qquad z \in (0,1]. \tag{14}$$

Setting $t = t_{n,i} = t_n + c_i h_n$ in Eq. (9) we have

$$\begin{aligned}\hat{\alpha}'_p(t_{n,i}) &= H_p(t_{n,i}) + f_p(t_{n,i})\hat{\alpha}_p(t_{n,i}) + \int_0^{t_n} k_p(t_{n,i},s)\hat{\alpha}_p(s)ds \\ &\quad + h_n \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n)\hat{\alpha}_p(t_n + sh_n)ds, \qquad \hat{\alpha}_p(0) = x_0\end{aligned}$$

by using (12), (13) and (14), it is written as follows:

$$\begin{aligned}&A_{n,i}^p + h_n f_p(t_{n,i}) \sum_{j=1}^{m} \mathsf{L}_j(c_i) A_{n,j}^p - h_n^2 \sum_{j=1}^{m} \left(\int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) \mathsf{L}_j(s) ds \right) A_{n,j}^p \\ &= H_p(t_{n,i}) + F_p^n(t_{n,i}) + \left(f_p(t_{n,i}) + h_n \int_0^{c_i} k_p(t_{n,i}, t_n + sh_n) ds \right) y_n,\end{aligned} \tag{15}$$

for $i = 1,\ldots,n$, where

$$F_p^n(t) = \int_0^{t_n} k_p(t,s)\hat{\alpha}_p(s)ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 k_p(t_{n,i}, t_\ell + sh_\ell)\hat{\alpha}_p(t_\ell + sh_\ell)ds, \tag{16}$$

Denotes the collocation solution on $\left[0,t_n\right]$. Using representation (14) and setting $t=t_{n,i}$ in (16) we get

$$F_p^n\left(t_{n,i}\right)=\sum_{\ell=0}^{n-1}h_\ell\left(\int_0^1 k_p\left(t_{n,i},t_\ell+sh_\ell\right)ds\right)y_\ell$$
$$+\sum_{\ell=0}^{n-1}h_\ell^2\sum_{j=1}^{m}\left(\int_0^1 k_p\left(t_{n,i},t_\ell+sh_\ell\right)\mathsf{L}_j\left(s\right)ds\right)A_{\ell,j}^p,$$

and by defining

$$B_p^{\ell,n}=\left(\int_0^1 k_p\left(t_{n,i},t_\ell+sh_\ell\right)\mathsf{L}_j\left(s\right)ds\right)_{i,j=1,\ldots,m}$$

for $0\le\ell<n\le N-1$, as a $\left(m\times m\right)$ matrix, has the form

$$F_p^n\left(t_{n,i}\right)=\sum_{\ell=0}^{n-1}h_\ell\left(\int_0^1 k_p\left(t_{n,i},t_\ell+sh_\ell\right)ds\right)y_\ell$$
$$+\sum_{\ell=0}^{n-1}h_\ell^2\sum_{j=1}^{m}\left[B_p^{\ell,n}\right]_{ij}A_{\ell,j}^p,\qquad i=1,\ldots,m$$

where $\left[B_p^{\ell,n}\right]_{ij}$ shows the (i,j)'th component of the matrix $B_p^{\ell,n}$. By letting

$$C_p^{\ell,n}=\left(\int_0^1 k_p\left(t_{n,1},t_\ell+sh_\ell\right)ds,\ldots,\int_0^1 k_p\left(t_{n,m},t_\ell+sh_\ell\right)ds\right)^T,$$
$$C_p^n=\left(\int_0^{c_1} k_p\left(t_{n,1},t_n+sh_n\right)ds,\ldots,\int_0^{c_m} k_p\left(t_{n,m},t_n+sh_n\right)ds\right)^T,$$
$$A_p^n=\left(A_{n,1}^p,\ldots,A_{n,m}^p\right)^T,$$
$$H_p^n=\left(H_p\left(t_{n,1}\right),\ldots,H_p\left(t_{n,m}\right)\right)^T,$$
$$f_p^n=\left(f_p\left(t_{n,1}\right),\ldots,f_p\left(t_{n,m}\right)\right)^T,$$
$$G_p^n=\left(F_p^n\left(t_{n,1}\right),\ldots,F_p^n\left(t_{n,m}\right)\right)^T,$$

and defining the $\left(m\times m\right)$ matrices

$$L_p^n=diag\left(f_p\left(t_{n,i}\right)\right)\left(\mathsf{L}_j\left(c_i\right)\right)_{i,j=1,\ldots,m},$$
$$B_p^n=\left(\int_0^{c_i} k_p\left(t_{n,i},t_n+sh_n\right)\mathsf{L}_j\left(s\right)ds\right)_{i,j=1,\ldots,m},$$

For $0\le n\le N-1$, the collocation equation (9) is reduced to the linear algebraic system

$$\left(I_m-h_n\left(L_p^n+h_nB_p^n\right)\right)A_p^n=H_p^n+G_p^n+\left(f_p^n+h_nC_p^n\right)y_n, \tag{17}$$

where $0\le n\le N$ and $p=1,2,\ldots$.

Here I_m denotes the $\left(m\times m\right)$ identity matrix. The existence and uniqueness of the collocation solution in $S_m^{(0)}\left(J_h\right)$ is considered in the following theorem.

Theorem 2. If $H_p(t)$, $f_p(t)$ and $k_p(t,s)$ in the Volterra integro-differential equation (9) are all continuous on their domains J and D, then there exists an $\bar{h}>0$ such that for any partition J_h with partition diameter h, $0<h<\bar{h}$, the linear algebraic system (17) has a unique solution A_p^n for $0 \le n \le N-1$ and $p=1,2,\ldots$.

Proof. The continuity of $H_p(t)$, $f_p(t)$ and $k_p(t,s)$ is obvious with respect to (7), (8) and Theorem 1. On the other hand the domains of $f_p(t)$ and $k_p(t,s)$ are compact, then the components of the matrices L_p^n and B_p^n for $0 \le n \le N-1$ and $p=1,2,\ldots$, are all bounded. These implies if h_n's are chosen sufficiently small, the inequality $h_n \left\| L_p^n + h_n B_p^n \right\| < 1$ holds and by Lemma 2 the inverse of the matrix $I_m - h_n\left(L_p^n + h_n B_p^n\right)$ exists. In other words, there is an $\bar{h}>0$ so that for any partition J_h with $h=\max\{h_n : 0 \le n \le N-1\} < \bar{h}$ the matrix $I_m - h_n\left(L_p^n + h_n B_p^n\right)$ has a uniformly bounded inverse and the proof is complete.

When the unknown vector A_p^n is computed from (17), the collocation solution for $t = t_n + z h_n \in \bar{\sigma}_n = [t_n, t_{n+1}]$ is given by

$$\hat{\alpha}_p(t_n + z h_n) = y_n + h_n \sum_{j=1}^{m} \mathrm{L}_j(z) A_{n,j}^p, \qquad z \in (0,1].$$

The convergence of this collocation solution is shown in the next theorem. The proof of this theorem with some changes may be found in [18].

Theorem 3. Suppose that in (6) ,$k_p(t,s) \in C^i[D,\square]$ and $H_p(t), f_p(t) \in C^i[J,\square]$, where $1 \le i \le m$, and $\hat{\alpha}_p \in S_m^{(0)}(J_h)$ is the collocation solution of equation (6) with $h \in (0,\bar{h})$. If $\alpha_p(t)$ is the exact solution of equation (6), then

$$\left\| \alpha_p - \hat{\alpha}_p \right\| \le C \left\| \alpha_p^{(i+1)} \right\| h^i,$$

holds on J, for any collocation points X_h. The constant C depends on the parameters $\{c_i\}$ but not on h.

The above argument yields an approximation solution $\hat{\alpha}_p(t)$ to the unique solution of the linear integro-differential equation (6) in the space $S_m^{(0)}(J_h)$ and the iterative scheme (3) or (6) produces a sequence $\{\alpha_p(t)\}$ that is quadratically convergent to the unique solution of nonlinear integro-differential equation(5). The inequality

$$\left\| x - \hat{\alpha}_p \right\| \le \left\| x - \alpha_p \right\| + \left\| \alpha_p - \hat{\alpha}_p \right\|, \tag{18}$$

and Theorems 1 and 3 show that the sequence of the collocation solutions $\{\hat{\alpha}_p(t)\}$ is convergent to the unique solution $x(t)$ of nonlinear equation (5). It is noticeable that in the relation (18) the first term is quadratically convergent and the convergence of the second term is $O(h^m)$.

5. Numerical Experiments

For numerical experiments, the presented method is applied to solve three different examples of the integro-differential equation (5). In each three examples the subintervals and the collocation parameters are chosen such that

$$h = h_n = \frac{T}{N}, \qquad n = 0,\dots,N-1.$$

$$c_i = \frac{i-1}{m-1}, \qquad i = 1,\dots,m.$$

Example 1. The first example is the following integro-differential equation:

$$x'(t) = x^5(t) + 1 - \frac{5}{4}t^5 + \int_0^t tsx^2(s)ds, \qquad x(0) = 0. \tag{19}$$

here $o \le t \le 1$, $k(t,s,x) = tsx^2$ and $f(t,s) = x^5 + 1 - \frac{5}{4}t^5$. Then k is nondecreasing and k and f are both convex with respect to x on D and $\alpha_0(t) = \frac{t}{2}$ is a lower solution of (19) on $[0,1]$. On the other hand, the exact solution is $x(t) = t$ where with respect to the Theorem 1 the solutions of the iterative scheme

$$\alpha_p'(t) = H_p(t) + 5\alpha_{p-1}^4(t)\alpha_p(t)$$
$$+2\int_0^t ts\alpha_{p-1}(s)\alpha_p(s)ds, \quad \alpha_p(0) = 0, \quad p = 1,2,\dots$$

with

$$H_p(t) = 1 - \frac{5}{4}t^5 - 4\alpha_{p-1}^5(t) - \int_0^t ts\alpha_{p-1}^2(s)ds,$$

is convergent to the exact solution of (19). To employ the given numerical procedure for approximating the solutions of these linear integro-differential equations, the values $m = 4, N = 5$ are chosen. The absolute values of errors, $|x(t_i) - \hat{\alpha}_p(t_i)|$, for equation (19) are shown in Table 1 and Figure 1 shows the convergence of the sequence $\{\hat{\alpha}_p(t)\}$ to the exact solution.

Table 1. Absolute errors in Example 1: $|x(t_i) - \hat{\alpha}_p(t_i)|$

$m = 4, N = 5$			
t_i	$p = 2$	$p = 4$	$p = 6$
0.125	2.5576E-12	5.5511E-15	5.5511E-17
0.250	1.0678E-11	7.2164E-14	1.1102E-17
0.375	6.5008E-08	8.1593E-14	2.2493E-16
0.500	2.5310E-07	9.4779E-13	2.8270E-16
0.625	1.8806E-05	6.2839E-12	3.0340E-16
0.750	1.1042E-04	2.1603E-12	5.6564E-15
0.875	1.4790E-04	1.4916E-11	7.1957E-15
1.000	9.4680E-03	3.5989E-10	7.1534E-15

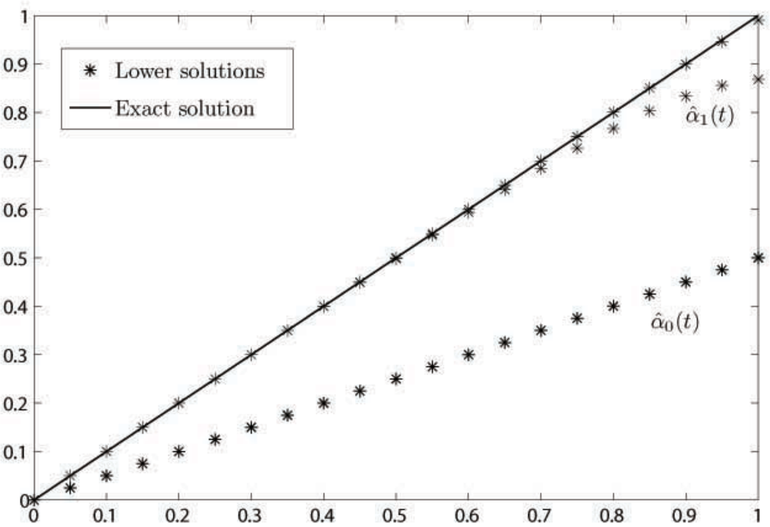

Fig 1. convergence of the sequence $\{\hat{\alpha}_p(t)\}$ to the exact solution of Example 1

Table 2. Absolute errors in Example 2 for lower solutions: $|x(t_i)-\hat{\alpha}_p(t_i)|$

$m=4, N=5$				$m=6, N=5$		
t_i	$p=2$	$p=4$	$p=6$	$p=2$	$p=4$	$p=6$
0.125	9.9528E-10	9.9528E-10	9.9528E-10	1.8496E-13	1.8995E-13	1.8995E-13
0.250	9.9683E-09	4.9761E-09	4.9761E-09	2.5706E-09	1.0480E-12	1.0480E-12
0.375	2.3559E-09	1.4969E-08	9.2451E-09	1.3277E-08	7.9847E-13	7.9847E-13
0.500	4.3242E-07	2.0257E-08	2.0257E-08	5.5749E-07	2.3450E-12	2.3450E-12
0.625	1.5111E-06	2.8473E-08	2.8473E-08	1.5322E-06	3.5107E-12	3.5107E-12
0.750	3.6688E-05	7.0239E-08	7.0248E-08	3.2160E-05	2.0534E-11	5.1030E-12
0.825	3.2453E-04	1.1222E-07	1.0102E-07	2.1509E-04	8.8511E-10	8.2789E-12
1.000	9.8331E-04	1.2696E-07	1.3180E-07	7.6373E-04	3.7158E-09	7.9523E-12

Table 3. Absolute errors in Example 3 for lower solutions: $|x(t_i)-\hat{\alpha}_p(t_i)|$

$m=5, N=5$				$m=6, N=5$		
t_i	$p=2$	$p=4$	$p=6$	$p=2$	$p=4$	$p=6$
0.125	1.3983E-10	1.3983E-10	1.3983E-10	1.7406E-12	1.7406E-12	1.7406E-12
0.250	1.7805E-10	1.7805E-10	1.7805E-10	4.6023E-11	4.6027E-12	4.6027E-12
0.375	1.6270E-10	1.6270E-10	1.6270E-10	3.3139E-11	3.3147E-12	3.3147E-12
0.500	5.9400E-11	3.4832E-11	3.4832E-11	2.2047E-10	5.1262E-11	5.1262E-12
0.625	1.8052E-11	8.1493E-11	8.1493E-11	1.0421E-10	5.3321E-11	5.3320E-12
0.750	3.4455E-08	5.1929E-11	5.1929E-11	2.4242E-08	4.8154E-11	4.8152E-12
0.825	5.6256E-07	2.1977E-11	2.1977E-11	4.5155E-07	4.8538E-10	4.8533E-12
1.000	3.0593E-06	9.6862E-11	9.6863E-11	2.2083E-06	2.6134E-10	2.6129E-11

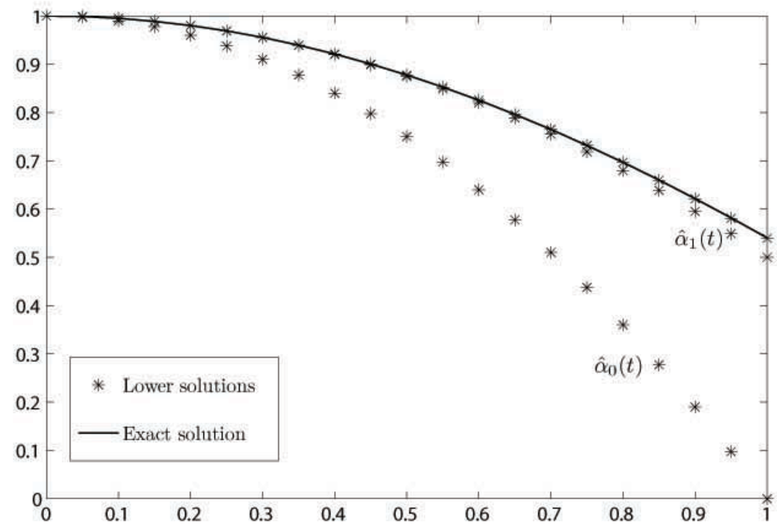

Fig 2. convergence of the sequence $\{\hat{\alpha}_p(t)\}$ to the exact solution of Example 2

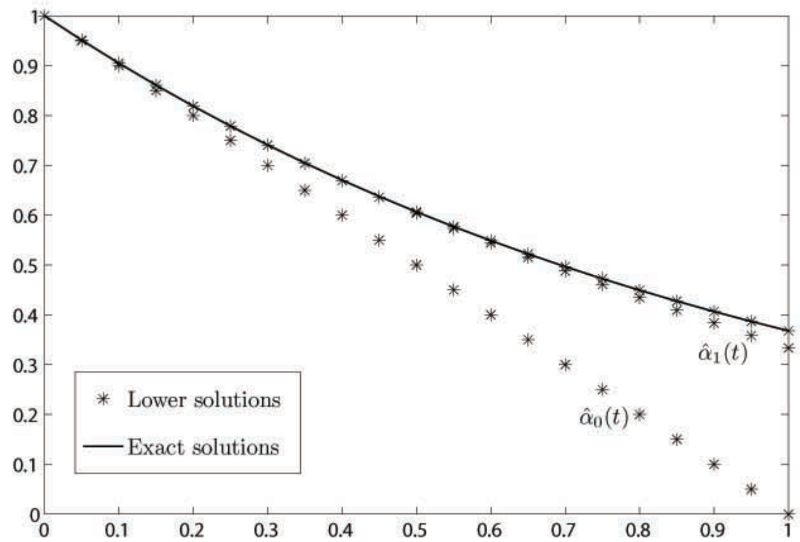

Fig 3. convergence of the sequence $\{\hat{\alpha}_p(t)\}$ to the exact solution of Example 3

Example 2. The second example

$$x'(t)=\frac{t}{5}\left(x^5(t)-1\right)-\sin(t)+\int_0^t t\sin(s)x^4(s)ds,$$

with initial condition $x(0)=1$ has the exact solution $x(t)=\cos(t)$ and the lower solution $\alpha_0(t)=1-t^2$ in the interval $[0,1]$. The kernel of this equation satisfies the necessary conditions and as in example 1 the method is applied to this equation and the absolute values of the errors are presented in Table 2 and Figure 2.

Example 3. As the third example, we have examined the method on the problem

$$x'(t)=t^2-\left(1+t^2+t^3\right)x(s)+\sin\left(t^2\right)-\sin\left(t^2x(t)\right)$$
$$-\int_0^t t^2x(s)\left(s+\cos\left(t^2x(s)\right)\right)ds, \qquad x(0)=1.$$

for $0\le t\le 1$, with the exact solution $x(t)=e^{-t}$. This problem is contained in the assumptions of the presented method with the lower solution $\alpha_0(t)=1-t$. Table 3 and Figure 3 show obtained results about this problem.

6. Conclusions

In this article we applied the method of quasilinearization and approximated the solution of nonlinear Volterra integro-differential equation. Collocation method was employed to solve the arisen linear integro-differential equations. Of advantage of the presented method is that we do not encounter solving nonlinear algebraic systems. Obtained numerical results show the accuracy and efficiency of the method. A weakness for this method is its limiting assumptions that a nonlinear equation must have them to be solvable by this method.

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