The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science / TJMCS Vol .4 No.4 (2012) 585-590

A Unique Common Fixed Point Theorem For Three Mappings in G –Cone metric spaces

K.P.R. Rao
Department of Mathematics
Acharya Nagarjuna University
Nagarjuna Nagar-522 510, A.P., India
email: kprrao2004@yahoo.com

K. Bhanu Lakshmi
Science and Humanities Department
Lakireddy Balireddy College of Engineering
Mylavaram-521230,A.P., India
email: bhanulaks@gmail.com

V.C.C. Raju
Department of Mathematics, University of Botswana
Private Bag UB 00704, Gaborone, Botswana
email: varanasi@mopipi.ub.bw

Received: March 2012 , Revised: November 2012
Online Publication: November 2012

Abstract

In this paper we obtain a unique common fixed point theorem for three mappings in G-cone metric spaces and obtain an extension and improvement of a theorem of I. Beg et. al. [1].

Keywords: G – cone metric space, common fixed points, symmetric space.

2010 Mathematics Subject Classification: 47H10; 54H25 .

1. Introduction and preliminaries

Based on cone metric spaces introduced by [2] and on G-metric spaces introduced by [4], I. Beg et. al. [1] introduced generalized cone metric spaces as follows:

Let E be a real Banach space and P be a subset of E. The subset P is called a Cone if it has the following properties:

(i) P is non empty, closed and $P \neq \{0\}$;

(ii) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in P^0$, where P^0 denotes the interior of P.

Proposition 1.1 ([5]). Let P be a cone in a real Banach space E . If $a \in P$ and $a \leq \lambda a$ for some $\lambda \in [0, 1)$ then $a = 0$.

Proposition 1.2 ([3],Cor.1.4). Let P be a cone in a real Banach space E .

(i) If $a \leq b$ and $b << c$, then $a << c$

(ii) If $a \in E$ and $a << c$ for all $c \in P^0$, then $a = 0$.

Remark 1.3 ([3]). $\lambda P^0 \subseteq P^0$ for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.

Definition 1.4 ([1]). Let X be a nonempty set and let $G : X \times X \times X \rightarrow E$ be a function satisfying the following properties :

(G_1): $G(x, y, z) = 0$ if $x = y = z$,

(G_2): $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_3): $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_4): $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$(symmetry in three variables),

(G_5): $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized cone metric on X and X is called a generalized cone metric space or a G – cone metric space. It is clear that if $G(x, y, z) = 0$ then $x = y = z$ for any $x, y, z \in X$.

Definition 1.5 ([1]). A G – cone metric space X is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 1.6 ([1]). Let X be a G – cone metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to converge to a point $x \in X$ if for every $c \in E$ with $0 << c$ there is N such that $G(x_n, x_m, x) << c$ for all $n, m > N$. In this case, we write $x_n \rightarrow x$ as $n \rightarrow \infty$.

The sequence $\{x_n\}$ is said to be a G – Cauchy sequence in X if for every $c \in E$ with $0 << c$ there is N such that $G(x_n, x_m, x_l) << c$ for all $n, m, l > N$.

X is said to be complete if every G – Cauchy sequence in X is convergent in X.

Proposition 1.7 ([1],Lemma 2.8). Let X be a G – cone metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent

(i) $\{x_n\}$ is G – convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 1.8 ([1],Lemma 2.9). Let X be a G – cone metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Remark 1.9 ([5]). If $c \in P^0$, $0 \le a_n$ and $a_n \to 0$, then there exists $n_0 \in N$ such that for all $n > n_0$ we have $a_n << c$.

Ismat Beg et.al [1] proved the following

Theorem 1.10 ([1],Theorem 3.1). Let X be a complete symmetric G – cone metric space and $T : X \to X$ be a mapping satisfying one of the following conditions

$$G(Tx, Ty, Tz) \le aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz)$$

and

$$G(Tx, Ty, Tz) \le aG(x, y, z) + bG(x, x, Tx) + cG(y, y, Ty) + dG(z, z, Tz)$$

for all $x, y, z \in X$, where $0 \le a + b + c + d < 1$.

Then T has a unique fixed point in X.

Now, we give a Lemma in G – cone metric spaces which is similar in cone metric spaces given by Jain et.al [6].

Lemma 1.11 : Let X be a G – cone metric space, P be a cone in a real Banach space E and $k_1, k_2, k_3, k_4 \ge 0$ such that $k_1 + k_2 + k_3 + k_4 > 0$ and $k > 0$. If $x_n \to x$, $y_n \to y$, $z_n \to z$ and $p_n \to p$ in X and

(1.11.1) $ka \le k_1G(x_n, x_m, x)+k_2G(y_n, y_m, y)+k_3G(z_n, z_m, z)+k_4G(p_n, p_m, p)$

then a = 0.

Proof. Since $x_n \to x$, $y_n \to y$, $z_n \to z$ and $p_n \to p$, we have for $c \in P^0$, there exists a positive integer N_c such that

$$\frac{c}{k_1+k_2+k_3+k_4} - G(x_n, x_m, x),\ \frac{c}{k_1+k_2+k_3+k_4} - G(y_n, y_m, y),$$

$$\frac{c}{k_1+k_2+k_3+k_4} - G(z_n, z_m, z),\ \frac{c}{k_1+k_2+k_3+k_4} - G(p_n, p_m, p) \in P^0 \forall\, n > N_c.$$

From Remark 1.3, we have

$$\frac{k_1 c}{k_1+k_2+k_3+k_4} - k_1 G(x_n, x_m, x),\ \frac{k_2 c}{k_1+k_2+k_3+k_4} - k_2 G(y_n, y_m, y),$$

$$\frac{k_3 c}{k_1+k_2+k_3+k_4} - k_3 G(z_n, z_m, z),\ \frac{k_4 c}{k_1+k_2+k_3+k_4} - k_4 G(p_n, p_m, p) \in P^0 \forall\, n > N_c.$$

Adding these four and by Remark 1.3, we have

$c - [k_1G(x_n, x_m, x)+k_2G(y_n, y_m, y)+k_3G(z_n, z_m, z)+k_4G(p_n, p_m, p)] \in P^0 \forall\ n > N_c$.

Now from(1.11.1) and Proposition 1.2(i), we have $ka \ll c$ for all $c \in P^0$.

By Proposition 1.2(ii), we have a = 0 as k > 0.

2. Main result

Theorem 2.1 . Let (X,G) be a symmetric G–cone metric space and A,B,C : X → X be satisfying

(2.1.1) $$G(Ax, By, Cz) \le k \max\left\{\begin{matrix} G(x, y, z), G(x, Ax, By), \\ G(y, By, Cz), G(z, Cz, Ax), \\ G(x, Ax, Ax), G(y, By, By), G(z, Cz, Cz), \end{matrix}\right\}$$

for all x, y, z ∈ X, where 0 ≤ k< 1.

Then the mappings A, B and C have a unique common fixed point in X.

Proof. Choose $x_0 \in X$. Define $x_{3n+1} = Ax_{3n}$, $x_{3n+2} = Bx_{3n+1}$, $x_{3n+3} = Cx_{3n+2}$, n = 0, 1, 2, ...

Case(I) If $x_{3n} = x_{3n+1}$ then x_{3n} is a fixed point of A. Denote x_{3n} = x. Then Ax = x.

Suppose Bx≠ Cx. Then from (2.1.1)

$$\begin{aligned} G(x, Bx, Cx) &= G(x, Bx, Cx) \\ &\le k \max\left\{\begin{matrix} 0, G(x, x, Bx), G(x, Bx, Cx), G(x, Cx, x), \\ 0, G(x, Bx, Bx), G(x, Cx, Cx) \end{matrix}\right\} \\ &= k \max \{G(x, x,Bx),G(x,Bx,Cx),G(x, x,Cx)\} \ ...(1) \text{ ,as X is symmetric} \\ &\le k\, G(x,Bx,Cx) \quad \text{from}(G_3) \end{aligned}$$

It is a contradiction. Hence Bx = Cx.

Now from(1),G(x,Bx,Bx) ≤ k G(x,Bx,Bx).

Now from Proposition 1.1, Bx = x. Hence Cx = x.

Thus x is a common fixed point of A,B and C.

Suppose x^1 is another common fixed point of A,B and C. Then

$$\begin{aligned} G(x, x, x^1) &= G(Ax,Bx,Cx^1) \\ &\le k \max \{G(x, x, x^1), 0,G(x, x, x^1),G(x^1, x^1, x), 0, 0, 0\} \\ &= k\, G(x, x, x^1) \text{ as X is symmetric} \end{aligned}$$

Hence $x = x^1$. Thus x is the unique common fixed point of A,B and C.

Similarly, if $x_{3n+1} = x_{3n+2}$ or $x_{3n+2} = x_{3n+3}$ then we can show that A , B and

C have a unique common fixed point in X.

Case(II): Assume that $x_n \ne x_{n+1}$ for all n.

As X is symmetric and from (G_3),we have

$$\begin{aligned} &G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &= G(Ax_{3n},Bx_{3n+1},Cx_{3n+2}) \\ &\le k \max\left\{\begin{matrix} G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ G(x_{3n+2}, x_{3n+3}, x_{3n+1}), G(x_{3n}, x_{3n+1}, x_{3n+1}) \\ G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \end{matrix}\right\} \end{aligned}$$

$$\leq \quad k \max \left\{ \begin{array}{c} G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ G(x_{3n}, x_{3n}, x_{3n+1}), G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \end{array} \right\}$$

$$\leq \quad k \max \left\{ \begin{array}{c} G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n}), G(x_{3n+2}, x_{3n+3}, x_{3n+1}) \end{array} \right\}$$

Thus $G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq k\, G(x_{3n}, x_{3n+1}, x_{3n+2})$.
Similarly, we can show that $G(x_{3n+2}, x_{3n+3}, x_{3n+4}) \leq k\, G(x_{3n+1}, x_{3n+2}, x_{3n+3})$
and $G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq G(x_{3n+2}, x_{3n+3}, x_{3n+4})$.
Thus $G(x_n, x_{n+1}, x_{n+2}) \leq k\, G(x_{n-1}, x_n, x_{n+1})$, n = 1, 2, 3, ...
Hence

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+2}) \leq\ & k\, G(x_{n-1}, x_n, x_{n+1}) \\ \leq\ & k^2 (G(x_{n-2}, x_{n-1}, x_n) \\ & . \\ & . \\ & . \\ \leq\ & k^n (G(x_0, x_1, x_2)) \quad ....(2) \end{aligned}$$

From (G_3) and (2), we have
$G(x_n, x_n, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+2}) \leq k^n(G(x_0, x_1, x_2))$.
Now for m > n,

$$\begin{aligned} G(x_n, x_n, x_m) \leq\ & G(x_n, x_n, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+2}) + \ldots + G(x_{m-1}, x_{m-1}, x_m) \\ \leq\ & k^n G(x_0, x_1, x_2) + k^{n+1} G(x_0, x_1, x_2) + \ldots + k^{m-1} G(x_0, x_1, x_2) \\ \leq\ & \frac{k^n}{1-k} G(x_0, x_1, x_2) \\ \rightarrow\ & 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

From Remark 1.9, it follows that for $0 << c$ and large n, $\frac{k^n}{1-k} G(x_0, x_1, x_2) << c$.

Now from Corollary 1.2(i), we have $G(x_n, x_n, x_m) << c$ for all m > n. Hence $\{x_n\}$ is G – Cauchy.
Since X is G – complete, there exists $p \in X$ such that $x_n \rightarrow p$ as $n \rightarrow \infty$.
Now

$$\begin{aligned} & G(Ap, p, p) \\ \leq\ & G(Ap, Bx_{3n+1}, Bx_{3n+1}) + G(Bx_{3n+1}, p, p) \\ \leq\ & G(Bx_{3n+1}, Cx_{3n+2}, Cx_{3n+2}) + G(Cx_{3n+2}, Ap, Bx_{3n+1}) + G(Bx_{3n+1}, p, p) \\ =\ & G(x_{3n+2}, x_{3n+3}, x_{3n+3}) + G(x_{3n+2}, p, p) + G(Ap, Bx_{3n+1}, Cx_{3n+2}) \\ \leq\ & G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) + G(x_{3n+2}, p, p) + \\ & k \max \left\{ \begin{array}{c} G(p, x_{3n+1}, x_{3n+2}), G(p, Ap, x_{3n+2}), \\ G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+2}, x_{3n+3}, Ap), \\ G(p, Ap, Ap), G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3}) \end{array} \right\} \\ \leq\ & 2\, G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) + \\ & k \max \left\{ \begin{array}{c} G(p, x_{3n+1}, x_{3n+2}), G(Ap, p, p) + G(p, p, x_{3n+2}), \\ G(x_{3n+2}, p, p) + G(p, x_{3n+1}, x_{3n+3}), G(Ap, p, p) + G(p, x_{3n+2}, x_{3n+3}), \\ G(p, p, Ap), G(x_{3n+2}, p, p) + G(p, x_{3n+1}, x_{3n+2}), \\ G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \end{array} \right\} \end{aligned}$$

Thus we have
$G(Ap, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) + k\, G(p, x_{3n+1}, x_{3n+2})$ or
$(1 - k)\, G(Ap, p, p) \leq (2 + k)G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3})$ or
$G(Ap, p, p) \leq (2+k)\, G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) + k\, G(p, x_{3n+1}, x_{3n+3})$ or
$(1-k)G(Ap, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) + kG(p, x_{3n+2}, x_{3n+3})$ or
$(1 - k)G(Ap, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3})$ or

$G(Ap, p, p) \leq (2+k)\ G(x_{3n+2}, p, p)+G(p, x_{3n+3}, x_{3n+3})+kG(p, x_{3n+1}, x_{3n+2})$ or
$G(Ap, p, p) \leq (2 + k)\ G(x_{3n+2}, p, p) + (1 + k)G(p, x_{3n+3}, x_{3n+3})$.
Now from Proposition 1.7 and from Lemma 1.11, it follows that$G(Ap, p, p) = 0$ so that $Ap = p$.
The rest of the proof follows as in Case(I).

References.

[1] Ismat Beg, Mujahid Abbas and Talat Nazir, Generalized cone metric spaces, J. Nonlinear Sci. Appl. 3, No.1 (2010), 21 – 31.

[2] L.G. Huang and X. Zhang , Cone metric spaces and fixed point theorems of contractive mappings, J.Math.Anal.Appl., 332(2) (2007), 1468 – 1476.

[3] S. Rezapour and R. Hamlbarani, Some notes on the paper " Cone metric spaces and fixed point theorems of contractive mappings ",J.Math.Anal. Appl., 345 (2008), 719-724.

[4] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis,Vol.7, no.2, (2006), 289 – 297.

[5] G. Jungck, S. Radenovic ,S. Radojevic and V. Rakocevic, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed point theory and Applications,Vol.2009,Article ID 643840,13 pages, doi:10.1155/2009/643840.

[6] Shoba Jain, Shishir Jain and Lal Bahadur, Compatibility and weak compatibility for four self maps in a cone metric space, Bulletin of Mathematical Analysis and Applications,Vol.2,Issue 1 (2010), 15 – 24.