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A £-FUZZY FIXED POINT THEOREM IN PARTIALLY ORDERED SETS AND APPLICATIONS

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ABSTRACT

An analogue of £-fuzzy Banach's fixed point theorem in partially ordered sets is proved in this paper, and several applications to linear and nonlinear matrix equations are discussed.

Keywords: £-Fuzzy contractive mapping; Complete £-fuzzy metric space; Fixed point theorem

1. Introduction

In this paper we discuss an analogue of $\mathcal{L}$-fuzzy Banach's fixed point theorem in partially ordered sets and several applications. The key feature in this fixed point theorem is that the contractility condition on the nonlinear map is only assumed to hold on elements that are comparable in the partial order. However, the map is assumed to be monotone. We show that under such conditions the conclusions of Banach's fixed point theorem still hold. It should be noted that there are many fixed point theorems for order-preserving or order-reversing maps on lattices. See [1]. However, for order-preserving maps the assumption is usually that the lattice is complete, which implies, for instance, that there is a maximal element in the lattice. Since the applications we have in mind concern the lattice of Hermitian matrices or the cone of positive definite matrices, this does not hold for some of our applications. For order-reversing maps there are usually conditions that imply that there cannot be a periodic orbit of period two. For our applications such assumptions are either not true or hard to check.

2. Preliminaries

In the sequel, we shall adopt the usual terminology, notation and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [9] which are a generalization of fuzzy metric sapces [4] and intuitionistic fuzzy metric spaces [7, 10].

Definition 2.1. *([5])* Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called a universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Lemma 2.2. ([2, 3]) *Consider the set L^* and the operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice .*

Classically, a triangular norm T on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = x$, for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 2.3. A negation on $\mathcal{L}$ is any strictly decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation.

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In this paper the negation $\mathcal{N} : L \to L$ is fixed.

Definition 2.4. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$; (boundary condition)

(ii) $(\forall(x,y) \in L^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$; (commutativity)
(iii) $(\forall(x,y,z) \in L^3)(\mathcal{T}(x,\mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$; (associativity)
(iv) $(\forall(x,x',y,y') \in L^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$. (monotonicity)

A t–norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y)$$

For example, $\mathcal{T}(x,y) = \min(x,y)$ and $\mathcal{T}(x,y) = xy$ are two continuous t–norms on $[0,1]$. A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n \in \mathbb{N})$ by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1,\cdots,x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1,\cdots,x_n),x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

A t-norm $\mathcal{T}$ is said to be *of Hadžić type* if the family $\{\mathcal{T}^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1_{\mathcal{L}}$, that is,

$$\forall\, \varepsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}\ \exists\, \delta \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\} : a >_L \mathcal{N}(\delta) \Rightarrow \mathcal{T}^n(a) >_L \mathcal{N}(\varepsilon) \quad (n \geq 1)\,.$$

$\mathcal{T}_M$ is a trivial example of a t-norm of Hadžić type, but there exist t-norms of Hadžić type weaker than $\mathcal{T}_M$ ([6]) where

$$\mathcal{T}_M(x,y) = \begin{cases} x, & \text{if } x \leq_L y, \\ y, & \text{if } y \leq_L x. \end{cases}$$

Definition 2.5. The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times\,]0,+\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0,+\infty[$:

(a) $\mathcal{M}(x,y,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x,y,t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t), \mathcal{M}(y,z,s)) \leq_L \mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,\cdot) :]0,\infty[\, \to L$ is continuous.

If the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfies the condition:

$$(f)\ \lim_{t\to\infty} \mathcal{M}(x,y,t) = 1_{\mathcal{L}},$$

then $(X, \mathcal{M}, \mathcal{T})$ is said to be *Menger $\mathcal{L}$-fuzzy metric space* or for short a $\mathbf{M}\mathcal{L}$-fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x,r,t) = \{y \in X : \mathcal{M}(x,y,t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $r \in L\setminus\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is called the *topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$*.

Example 2.6. ([11]) Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ be defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \left(\frac{t}{t+d(x,y)}, \frac{d(x,y)}{t+d(x,y)}\right).$$

Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 2.7. Let $X = \mathbb{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^*, and let $\mathcal{M}(x, y, t)$ on $X^2 \times (0, \infty)$ be defined as follows:

$$\mathcal{M}(x,y,t) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & if \quad x \le y \\ (\frac{y}{x}, \frac{x-y}{x}) & if \quad y \le x. \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, \mathcal{T})$ is an $\mathcal{L}$-fuzzy metric space.

Lemma 2.8. ([4]) *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Definition 2.9. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a *Cauchy sequence*, if for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \ge n \ge n_0$ $(n \ge m \ge n_0)$,

$$\mathcal{M}(x_m, x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ whenever $n \to +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent.

Let $\mathcal{T}$ be a continuous t–norm on the lattice $\mathcal{L}$ such that for every $\mu \in L\setminus\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, there is a $\lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ (which may depend on n) such that

$$(2.1) \qquad \mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu) \ \text{ for each } \ n \in \{1, 2, ...\}.$$

For the remainder of this paper we assume (1.1) holds.

Definition 2.10. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0, \infty[$ converges to a point $(x, y, t) \in X \times X \times]0, \infty[$ i.e., $\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_{\mathcal{L}}$ and $\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 2.11. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X \times X \times]0, \infty[$.*

Proof. The proof is the same as that for fuzzy spaces (see Proposition 1 of [?]).

Lemma 2.12. *If a $\mathrm{M}\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfies the following condition:*

$$\mathcal{M}(x, y, t) = C, \quad \textit{for all } t > 0.$$

Then we have $C = 1_{\mathcal{L}}$ and $x = y$.

Proof. Let $\mathcal{M}(x, y, t) = C$ for all $t > 0$. Then by (f) of Definition 1.8, we have $C = 1_{\mathcal{L}}$ and by (b) of Definition 1.5 we conclude that $x = y$.

Lemma 2.13. ([6]) *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathrm{M}\mathcal{L}$-fuzzy metric space which $\mathcal{T}$ is Hadžić type. Suppose*

$$\mathcal{M}(x_n, x_{n+1}, t) \geq_L \mathcal{M}\left(x_0, x_1, \frac{t}{k^n}\right)$$

for some $0 < k < 1$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

3. Fixed point theorem

Theorem 3.1. *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathrm{M}\mathcal{L}$-fuzzy metric space, in which $\mathcal{T}$ is Hadžić type. If A is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that*

(1) $\exists\, 0 < c < 1 : \mathcal{M}(Fx, Fy, t) \geq_L \mathcal{M}\left(x, y, \frac{t}{c}\right)$, $\forall x \geq y \in X$;

(2) $\exists\, x_0 \in X$: $x_0 \leq F(x_0)$ *or* $x_0 \geq F(x_0)$,

then F has a unique fixed point $\bar{x}$. Moreover, for every $x \in X$,

$$\lim_{n \to \infty} F^n(x) = \bar{x}.$$

Proof. Let $x_0 \in X$ be such that $x_0 \leq F(x_0)$ or $x_0 \geq F(x_0)$. The monotonicity of F implies that either $F^n(x_0) \leq F^{n+1}(x_0)$ or $F^n(x_0) \geq F^{n+1}(x_0)$ for $n = 1, 2, 3, \ldots$. So,

$$\mathcal{M}(F^{n+1}(x_0), F^n(x_0), t) \geq \mathcal{M}\left(F^n(x_0), F^{n-1}(x_0), \frac{t}{c}\right).$$

Hence, induction gives

$$\mathcal{M}(F^{n+1}(x_0), F^n(x_0), t) \geq \mathcal{M}\left(F(x_0), x_0, \frac{t}{c^n}\right).$$

By Lemma 2.13, $\{F^n(x_0)\}$ is Cauchy sequence. Since F is complete, it follows that,

$$\lim_{n \to \infty} F^n(x_0) = \bar{x}.$$

Also, $\bar{x}$ is the unique fixed point of F see [8].

Remark 3.2. There are some application from fixed point theorem in $\mathrm{M}\mathcal{L}$-fuzzy metric space to linear and nonlinear matrix equations. In fact, by a standard $\mathrm{M}\mathcal{L}$-fuzzy metric (see Example 2.6) we can conclude all result of [8].

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