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Fixed Point and Hyers-Ulam-Rassias Stability of a Quadratic Functional Equation in Menger Probabilistic Normed Spaces

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Abstract

In this paper, using the fixed point alternative approach, we investigate the Hyers Ulam-Rassias stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

in Menger probabilistic normed spaces.

Keywords: fixed point theory, Hyers-Ulam-Rassias stability.

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1. Introduction.

In [1] K. Menger proposed the probabilistic concept of distance by replacing the number $d(p,q)$, as distance between points (p,q) by a distribution function $F_{(p,q)}$. This idea led to a large development of probabilistic analysis [1],[2].

Probabilistic normed spaces were first defined by Serstnev in [3]. So, a fruitful theory concordant with that of ordinary normed spaces and with that of probabilistic metric spaces was initiated. The theory of probabilistic normed spaces is important as a random generalization of deterministic linear normed space theory. in the same time it gives also new tools in the study of random operator equations. For important results of probabilistic functional analysis we refer to [1],[2],[4].

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? If the problem accepts a solution, we say that the equation is stable. The first stability problem concerning group homomorphisms was raised by Ulam [5] in 1940. In the next year D.H. Hyres [6], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

Subsequently the result of Hyers was generalized by Aoki [8] for additive mapping and by Th. M. Rassias [4] for linear mapping by considering an unbounded cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call the Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called *a quadratic functional equation* . In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [8] for mappings $f:X\longrightarrow Y$ where X is a normed space and Y is a Banach space.

In 1996, G. Isac and Th. M. Rassias [6] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Some authors considered the stability of quadratic functional equation random normed space [22] . By fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors(see [9],[10],[11]).

Definition 1.1. A function $f:R\to[0,1]$ is called adistribution function if it is nondecreasing and left-continuous, with $\sup_{t\in R}F(t)=1$ and $\inf_{t\in R}F(t)=0$.

The class of all distribution functions F with $F(0)=0$ is denoted by D_+. ε_0 is the element of D_+ define by

$$\varepsilon_0=\begin{cases}1 & t>0\\ 0 & t\le 0\end{cases}$$

Definition 1.2. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a t-norm if it satisfies the following condition:

(1) * is commutative and associative;

(2) * is continuous;

(3) $a*1=a$ for all $a\in[0,1]$;

(4) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition1.3. [13] Let X be a real vector space, F a mapping from X to D_+ (for any $x\in X$, $F(X)$, is denoted by F_x) and * a t-norm. The triple $(X,F,*)$ is called a Menger probabilistic normd spaces (briefly Menger PN-space) if the following conditions are satisfied:

(1) $F_x(0)=0$, for all $x\in X$;

(2) $F_x(0)=\varepsilon_0$ iff $x=\theta$;

(3) $F_{\alpha x}(t)=F_x(\frac{t}{|\alpha|})$ for all $\alpha\in R,\alpha\ne 0$ and $x\in X$;

(4) $F_{x+y}(t_1+t_2)\ge F_x(t_1)*F_y(t_2)$ for all $x,y\in X$ and t_1,t_2 >0.

Definition 1.4. Let $(X,F,*)$ be a Menger PN-space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that

$$\lim_{n\to\infty} F_{x_n - x}(t) = 1$$

For all $t > 0$. In this case, x is called the limit of $\{x_n\}$.

Definition 1.5. The sequence $\{x_n\}$ in Menger PN-space $(X, F, *)$ is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$ there exists some n_0 such that $F_{x_n - x_m}(\delta) > 1 - \varepsilon$ for all $m, n > n_0$.
Clearly, every convergent sequence in Menger PN-space is Cauchy. If each Cauchy sequence is convergent sequence in a Menger PN-space $(X, F, *)$, then $(X, F, *)$ is called Menger probabilistic Banach space (briefly, Menger PB-space).

Definition 1.6. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if satisfies the following conditions:
(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$;
Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 1.1. Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant L <1. Then for all $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X ; d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \dfrac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

2. Main results

Throughout this section, using fixed point method, we prove the Hyers-Ulam-Rassias stability of quadratic functional equation in Menger probabilistic normed spaces.
Definition 2.1. Let $(X, F, *)$ be a Menger PN-spaces and $(Y, G, *)$ be a menger PB-spaces. A mapping $f : X \to Y$ is said to be P-approximately quadratic if

$$G_{f(x+y)+f(x-y)-2f(x)-2f(y)}(t+s) \ge F_x(t) * F_y(s) \tag{2.2}$$

for all $t, s > 0$.
Theorem 2.1. Let $f : X \to Y$ be a P-approximately quadratic functional equation and there exists $0 < \alpha < \dfrac{1}{4}$ such that

$$F_x(2t) \ge F_{\alpha x}(t) \tag{2.3}$$

Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$G_{f(x)-Q(x)}(t) \ge F_x\left(\frac{1-4\alpha}{2\alpha} t\right) \tag{2.4}$$

Proof. Puttin $x = y$ and $s = t$ in (2.2), we have

$$G_{f(2x)-4f(x)}(2t) \geq F_x(t) \tag{2.5}$$

Replacing x by $\frac{x}{2}$ in (2.5), we have

$$G_{f(x)-4f(\frac{x}{2})}(2t) \geq F_{\frac{x}{2}}(t). \tag{2.6}$$

With the definition (1.3) and replacing t by $\frac{t}{2}$ in (2.6) we obtain

$$G_{f(x)-4f(\frac{x}{2})}(t) \geq F_x(t) \tag{2.7}$$

for all $x \in X$, \$t>0\$. Consider the set $K := \{g : X \to Y : g(0) = 0\}$ and the generalized metric in K defined by

$$d(h,g) = \inf\left\{c \in [0,\infty] \mid G_{g(x)-h(x)}(ct) \geq F_x(t)\right\}. \tag{2.8}$$

Where $\inf \varnothing = +\infty$. It is easy to show that (K,d) is complete (see [22], lemma 2.1.). Now, we consider a linear mapping $J : K \to K$ such that $Jh(x) = 4h(\frac{x}{2})$ for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant 4α. In fact let $g,h \in K$ be such that $d(g,h) < c$. Then we have

$$G_{g(x)-h(x)}(ct) \geq F_x(t) \tag{2.9}$$

Whence

$$\begin{aligned} G_{Jg(x)-Jh(x)}(4\alpha ct) &= G_{4g(\frac{x}{2})-4h(\frac{x}{2})}(4\alpha ct) \\ &= G_{g(\frac{x}{2})-h(\frac{x}{2})}(\alpha ct) \\ &\geq F_{\frac{x}{2}}(\alpha t) \\ &\geq F_x(t) \end{aligned} \tag{2.10}$$

For all $x \in X$, $t > 0$. Then

$$d(Jg, Jh) < 4\alpha c.$$

This mean that

$$d(Jg, Jh) \leq 4\alpha d(g,h) \tag{2.11}$$

for all $g,h \in K$. It follows from (2.7) that

$$d(f, Jf) \leq 2\alpha \tag{2.12}$$

Now, by Theorem (1.1) there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J , that is ,

$$Q(\frac{x}{2}) = \frac{1}{4}Q(x) \tag{2.13}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$\Omega = \{h \in K : d(g,h) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists $c \in [0,\infty]$ satisfying

$$G_{f(x)-Q(x)}(ct) \geq F_x(t) \tag{2.14}$$

for all $x \in X$ and $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f(\frac{x}{2^n}) = Q(x) \tag{2.15}$$

for all $x \in X$.

(3) $d(f,Q) \le \frac{d(f,Jf)}{1-L} \le \frac{2\alpha}{1-4\alpha}$ with $f \in \Omega$ and so

$$G_{f(x)-Q(x)}\left(\frac{2\alpha t}{1-4\alpha}\right) \ge F_x(t). \tag{2.16}$$

This implies that

$$G_{f(x)-Q(x)}(t) \ge F_x\left(\frac{1-4\alpha}{2\alpha}\right). \tag{2.17}$$

Then the inequality (2.4) holds. On the other hand

$$G_{4^n\left[f(\frac{x}{2^n}+\frac{y}{2^n})+f(\frac{x}{2^n}-\frac{y}{2^n})-2f(\frac{x}{2^n})-2f(\frac{y}{2^n})\right]}(t+s) \ge F_{\frac{x}{2^n}}(t) * F_{\frac{y}{2^n}}(s) \tag{2.18}$$

By definition (1.3) and (2.3) we have

$$F_{\frac{x}{2^n}}(t) \ge F_{\alpha^n x}(t) \qquad F_{\frac{y}{2^n}}(s) \ge F_{\alpha^n y}(s)$$

So

$$F_{\frac{x}{2^n}}(t) * F_{\frac{y}{2^n}}(s) \ge F_{\alpha^n x}(t) * F_{\alpha^n y}(s) \tag{2.19}$$

for all $x, y \in X$ and $t, s > 0$. Now, since $\lim_{n\to\infty} F_{\alpha^n x}(t) * F_{\alpha^n y}(s) = 1$ we have

$$G_{Q(x+y)+Q(x-y)-2Q(x)-2Q(y)}(t+s) = 1 \tag{2.20}$$

for all $x, y \in X$. This complete the proof. ■

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