

## **Solution of systems of integral - differential equations by Variational Iteration Method**

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### **Abstract.**

In this paper, we will consider variational iteration method (VIM) for solving systems of integral–differential equation. This method is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. Using the variational iteration method, it is possible to find the exact solution or an approximate solution of the problem. In this paper, variational iteration method is introduced to overcome the difficulty arising in calculating Adomian polynomials.

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**Key words:** . Variational iteration method; Systems of integral–differential equations.

### 1. Introduction

A system of integral–differential equations can be considered, in general form, as:

$$\begin{cases} y_1'(t) = f_1(t, y_1(t), y_2(t), \dots, y_m(t), \int_0^t g_1(t, y_1(s), y_2(s), \dots, y_m(s)) ds), & y_1(0) = a_1, \\ y_2'(t) = f_2(t, y_1(t), y_2(t), \dots, y_m(t), \int_0^t g_2(t, y_1(s), y_2(s), \dots, y_m(s)) ds), & y_2(0) = a_2, \\ \vdots \\ y_m'(t) = f_m(t, y_1(t), y_2(t), \dots, y_m(t), \int_0^t g_m(t, y_1(s), y_2(s), \dots, y_m(s)) ds), & y_m(0) = a_m \end{cases} \quad (1)$$

where each equation represents the first derivative of one of the unknown functions as a mapping involving the independent variable  $t$ , and  $n$  unknown functions  $f_1, f_2, \dots, f_m$  which have appeared partly in the integral sign.

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In this paper, we use variational iteration method for solving system (1). This method was first proposed by J. He [7,8], and systematically illustrated in [5].

### 2. Variational iteration method

In this section, we use the following non-homogeneous system of differential equations

$$\begin{cases} L_1 u_1(t) + N_1(u_1(t), u_2(t), \dots, u_m(t)) = g_1(t), \\ L_2 u_2(t) + N_2(u_1(t), u_2(t), \dots, u_m(t)) = g_2(t), \\ \vdots \\ L_m u_m(t) + N_m(u_1(t), u_2(t), \dots, u_m(t)) = g_m(t) \end{cases} \quad (2)$$

to illustrate the basic idea of the variational iteration method. In the above system of equations  $L_1, L_2, \dots, L_m$  are linear differential operators with respect to  $t$  and  $N_1, N_2, \dots, N_m$  are nonlinear operators and  $g_1(t), g_2(t), \dots, g_m(t)$  are some given functions. According to the variational iteration method, we can construct a correct functional as follows :

$$\begin{cases} u_{1_{n+1}}(t) = u_{1_n}(t) + \int_0^t \lambda_1(\tau) \{L_1 u_{1_n}(\tau) + N_1(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_1(\tau)\} d\tau, \\ u_{2_{n+1}}(t) = u_{2_n}(t) + \int_0^t \lambda_2(\tau) \{L_2 u_{2_n}(\tau) + N_2(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_2(\tau)\} d\tau, \\ \vdots \\ u_{m_{n+1}}(t) = u_{m_n}(t) + \int_0^t \lambda_m(\tau) \{L_m u_{m_n}(\tau) + N_m(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_m(\tau)\} d\tau, \end{cases} \quad (3)$$

where  $\lambda_1(\tau), \lambda_2(\tau), \dots, \lambda_m(\tau)$  are general Lagrange multipliers, which can be identified optimally via variational theory. The second terms on the right-hand side in (3) are called the correction and the subscript  $n$  denotes the  $n$ th order approximation. Under a suitable

restricted variational assumptions (i.e.  $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m$ ), then the Lagrange multipliers are identified. Now we may start the procedures with the given initial approximation and using the above iteration formulas to obtain the approximate solutions. For more see [1], [3-6] and [11-13].

We should now point that after obtaining value  $\lambda_1(\tau), \lambda_2(\tau), \dots, \lambda_m(\tau)$  it is better to use below iteration formulas (4) instead of above iteration formulas (2) for obtaining better and closer approximations to the exact solution.

$$\begin{cases} u_{1_{n+1}}(t) = u_{1_n}(t) + \int_0^t \lambda_1(\tau) \{L_1 u_{1_n}(\tau) + N_1(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_1(\tau)\} d\tau, \\ u_{2_{n+1}}(t) = u_{2_n}(t) + \int_0^t \lambda_2(\tau) \{L_2 u_{2_n}(\tau) + N_2(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_2(\tau)\} d\tau, \\ u_{3_{n+1}}(t) = u_{3_n}(t) + \int_0^t \lambda_3(\tau) \{L_3 u_{3_n}(\tau) + N_3(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_3(\tau)\} d\tau, \\ \vdots \\ u_{m_{n+1}}(t) = u_{m_n}(t) + \int_0^t \lambda_m(\tau) \{L_m u_{m_n}(\tau) + N_m(\tilde{u}_{1_n}, \tilde{u}_{2_n}, \dots, \tilde{u}_{m_n}) - g_m(\tau)\} d\tau, \end{cases} \quad (4)$$

The iteration formula (4) will give several approximations of  $u_1(\tau), u_2(\tau), \dots, u_m(\tau)$  and the exact solutions are obtained at the limit of the resulting successive approximations .i.e.

$$\begin{aligned} u_1(t) &= \lim_{n \rightarrow \infty} u_{1_n}(t) \\ u_2(t) &= \lim_{n \rightarrow \infty} u_{2_n}(t) \\ &\vdots \\ u_m(t) &= \lim_{n \rightarrow \infty} u_{m_n}(t) \end{aligned}$$

### 3. Test problems

In this section, we present four examples. These examples are considered to illustrate the method for linear and nonlinear system of ordinary integral-differential equations.

**Example 1.** In this example the following system of integral–differential equations is solved, the exact solution are  $y_1(x) = x + e^x, y_2(x) = x - e^x$ .

$$\begin{cases} y_1' = 1 + x + x^2 - y_2 - \int_0^x (y_1(t) + y_2(t)) dt, & y_1(0) = 1; \\ y_2' = -1 - x - y_1 - \int_0^x (y_1(t) + y_2(t)) dt, & y_2(0) = -1. \end{cases} \quad (5)$$

According to the variational iteration method we consider the correction functional in the following form (see [1, 3–6, 11–13]):

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \lambda_1(\tau) \{y_{1_n}' + \tilde{y}_{2_n} - \int_0^\tau \tilde{y}_{1_n}(s) + \tilde{y}_{2_n}(s) ds - 1 - \tau - \tau^2\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \lambda_2(\tau) \{y_{2_n}' - \tilde{y}_{1_n} + \int_0^\tau \tilde{y}_{1_n}(s) - \tilde{y}_{2_n}(s) ds + 1 + \tau\} d\tau, \end{cases} \quad (6)$$

where  $\lambda_1(\tau)$  and  $\lambda_2(\tau)$  are the general Lagrange multipliers which its optimal value is found by using variational theory. the value of  $y_{1_0}$  and  $y_{2_0}$  are initial approximations and must be chosen suitably and  $\tilde{y}_{1_0}$  and  $\tilde{y}_{2_0}$  are the restricted values i.e.  $\delta y_{1_0} = 0$  and  $\delta y_{2_0} = 0$ . Now we have

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau) \{y'_{1_n} + \tilde{y}_{2_n} - \int_0^\tau \tilde{y}_{1_n}(s) + \tilde{y}_{2_n}(s) ds - 1 - \tau - \tau^2\} d\tau, \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau) \{y'_{2_n} - \tilde{y}_{1_n} + \int_0^\tau \tilde{y}_{1_n}(s) - \tilde{y}_{2_n}(s) ds + 1 + \tau\} d\tau, \end{cases} \quad (7)$$

to find the optimal value of  $\lambda_i$

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau) \{y'_{1_n}\} d\tau = \delta y_{1_n}(x) + \delta \lambda_1(\tau) y_{1_n}(\tau) |_{\tau=x} - \delta \int_0^x \lambda'_1(\tau) y_{1_n}(\tau) d\tau, \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau) \{y'_{2_n}\} d\tau = \delta y_{2_n}(x) + \delta \lambda_2(\tau) y_{2_n}(\tau) |_{\tau=x} - \delta \int_0^x \lambda'_2(\tau) y_{2_n}(\tau) d\tau, \end{cases} \quad (8)$$

and this yields to the following stationary conditions:

$$(9) \quad 1 + \lambda_1(\tau) = 0|_{\tau=x}, \quad \lambda'_1(\tau) = 0|_{\tau=x},$$

$$(10) \quad 1 + \lambda_2(\tau) = 0|_{\tau=x}, \quad \lambda'_2(\tau) = 0|_{\tau=x}.$$

Therefore the Lagrange multipliers is defined as the following form:

$$(11) \quad \lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1$$

substituting Eq. (11) into the correctional functional Eq. (6) we get the following iteration formula :

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \{y'_{1_n}(\tau) + y_{2_n}(\tau) - \int_0^\tau (y_{1_n}(s) + y_{2_n}(s)) ds - 1 - \tau - \tau^2\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \{y'_{2_n}(\tau) + y_{1_n}(\tau) - \int_0^\tau (y_{1_n}(s) + y_{2_n}(s)) ds + 1 + \tau\} d\tau. \end{cases} \quad (12)$$

As say before, it is better to use below iteration formulas (13) instead of above iteration formulas (12) for obtaining better and closer approximations to the exact solution.

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \{y'_{1_n}(\tau) + y_{2_n}(\tau) - \int_0^\tau (y_{1_n}(s) + y_{2_n}(s)) ds - 1 - \tau - \tau^2\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \{y'_{2_n}(\tau) + y_{1_{n+1}}(\tau) - \int_0^\tau (y_{1_{n+1}}(s) + y_{2_n}(s)) ds + 1 + \tau\} d\tau \end{cases} \quad (13)$$

We start with initial approximations  $y_{1_0}(x) = y_1(0)$  and  $y_{2_0}(x) = y_2(0)$ , then by the iteration formulas (13) and MATLAB software we have

$$y_{1_1}(x) = 1 + 2x + \frac{1}{2}x^2 + \left(\frac{1}{3}x^3\right),$$

$$y_{2_1}(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \left(-\frac{1}{60}x^5\right),$$

$$y_{1_2}(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \left(-\frac{1}{60}x^5 + \frac{1}{720}x^6 + \frac{1}{2520}x^7\right),$$

$$y_{2_2}(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 + \left(-\frac{1}{360}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 - \frac{1}{181440}x^9\right),$$

$$y_{1_3}(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{60}x^5 + \frac{1}{720}x^6 + \left(\frac{1}{1008}x^7 - \frac{1}{90720}x^9 + \dots\right),$$

$$y_{2_3}(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \left(\frac{1}{20160}x^8 - \frac{1}{90720}x^9 + \dots\right),$$

$$y_{1_4}(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \left(-\frac{1}{60480}x^9 + \frac{1}{1814400}x^{10} + \dots\right),$$

$$y_{2_4}(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 - \frac{1}{40320}x^8 - \frac{1}{362880}x^9 + \left(-\frac{1}{725760}x^{10} + \frac{1}{9979200}x^{11}\right)$$

⋮

and so on. Thus, we obtain

$$y_1(x) = \lim_{n \rightarrow \infty} y_{1_n}(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + \dots,$$

$$y_2(x) = \lim_{n \rightarrow \infty} y_{2_n}(x) = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 - \frac{1}{5040}x^7 - \frac{1}{40320}x^8 - \frac{1}{362880}x^9 - \frac{1}{3628800}x^{10} + \dots,$$

which are quite close to the Taylor series of their exact solutions  $y_1(x) = x + e^x$  and  $y_2(x) = x - e^x$ .

**Example 2.** In this example the following system of integral–differential equations is solved, the exact solution are  $y_1(x) = \sin(x)$ ,  $y_2(x) = \cos(x)$ .

$$(14) \quad \begin{cases} y_1' = -x + y_2 + \int_0^x (y_1^2(t) + y_2^2(t))dt, & y_1(0) = 0; \\ y_2' = -x + y_1 + \int_0^x (y_1^2(t) + y_2^2(t))dt, & y_2(0) = 1. \end{cases}$$

According to the variational iteration method we consider the correction functional in the following form (see [1, 3–6, 11–13]):

$$(15) \quad \begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \lambda_1(\tau) \left\{ y_{1_n}' - \tilde{y}_{2_n} - \int_0^\tau (\tilde{y}_{1_n}^2(s) + \tilde{y}_{2_n}^2(s)) ds + \tau \right\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \lambda_2(\tau) \left\{ y_{2_n}' + \tilde{y}_{1_n} + \int_0^\tau (\tilde{y}_{1_n}^2(s) + \tilde{y}_{2_n}^2(s)) ds - \tau \right\} d\tau, \end{cases}$$

Where  $\lambda_1(\tau)$  and  $\lambda_2(\tau)$  are the general Lagrange multipliers which its optimal value is found by using variational theory. The value of  $y_{1_0}$  and  $y_{2_0}$  are initial approximations and must be chosen suitably and  $\tilde{y}_{1_n}$  and  $\tilde{y}_{2_n}$  are the restricted values i.e.  $\delta \tilde{y}_{1_n} = 0$  and  $\delta y_{2_n} = 0$ . As previous example, we can obtain stationary conditions in the following form:

$$(16) \quad 1 + \lambda_1(\tau) = 0|_{\tau=x}, \quad \lambda_1'(\tau) = 0|_{\tau=x},$$

$$(17) \quad 1 + \lambda_2(\tau) = 0|_{\tau=x}, \quad \lambda_2'(\tau) = 0|_{\tau=x},$$

Therefore the Lagrange multipliers is defined as the following form:

$$(18) \quad \lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1 .$$

Substituting Eq. (18) into the correctional functional Eq. (16) we get the following iteration formula:

$$(19) \quad \begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) - \int_0^x \left\{ y_{1_n}' - y_{2_n} - \int_0^\tau (y_{1_n}^2(s) + y_{2_n}^2(s)) ds + \tau \right\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) - \int_0^x \left\{ y_{2_n}' + y_{1_n} + \int_0^\tau (y_{1_n}^2(s) + y_{2_n}^2(s)) ds - \tau \right\} d\tau, \end{cases}$$

As say before, it is better to use below iteration formulas (20) instead of above iteration formulas (19) for obtaining better and closer approximations to the exact solution.

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) - \int_0^x \{y'_{1_n} - y_{2_n} - \int_0^\tau (y_{1_n}^2(s) + y_{2_n}^2(s)) ds + \tau\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) - \int_0^x \{y'_{2_n} + y_{1_{n+1}} + \int_0^\tau (y_{1_{n+1}}^2(s) + y_{2_n}^2(s)) ds - \tau\} d\tau. \end{cases} \quad (20)$$

We start with initial approximations  $y_{1_0}(x) = y_1(0)$  and  $y_{2_0}(x) = y_2(0)$ , then by the iteration formulas (20) and MATLAB software we have

$$y_{1_1}(x) = x,$$

$$y_{2_1}(x) = 1 - \frac{1}{2}x^2 + \left(-\frac{1}{12}x^4\right),$$

$$y_{1_2}(x) = x - \frac{1}{6}x^3 + \left(-\frac{1}{60}x^5 + \frac{1}{360}x^6 + \frac{1}{672}x^8 + \frac{1}{12960}x^{10}\right),$$

$$y_{2_2}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \left(-\frac{1}{2520}x^7 - \frac{1}{720}x^8 - \frac{11}{45360}x^9 + \frac{1}{12960}x^{10} - \frac{4}{155925}x^{11} + \dots\right),$$

$$y_{1_3}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \left(\frac{1}{360}x^7 - \frac{1}{2016}x^8 - \frac{1}{11340}x^9 - \frac{59}{6604800}x^{10} + \frac{13}{2494800}x^{11} + \dots\right),$$

$$y_{2_3}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \left(-\frac{13}{20160}x^8 + \frac{1}{15120}x^9 + \frac{253}{1814400}x^{10} + \frac{1}{19958400}x^{11} + \dots\right),$$

$$y_{1_4}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \left(-\frac{13}{181440}x^9 + \frac{10}{604800}x^{10} + \frac{97}{19958400}x^{11} + \dots\right),$$

$$y_{2_4}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 + \left(\frac{1}{45360}x^{10} - \frac{1}{246400}x^{11} - \frac{1549}{239500800}x^{12} + \dots\right),$$

⋮

and so on. Thus, we obtain

$$y_1(x) = \lim_{n \rightarrow \infty} y_{1_n}(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{479001600}x^{12} + \dots,$$

$$y_2(x) = \lim_{n \rightarrow \infty} y_{2_n}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} + \frac{1}{4790001600}x^{12} + \dots,$$

which are quite close to the Taylor series of their exact solutions  $y_1(x) = \sin(x)$  and

$$y_2(x) = \cos(x).$$

**Example 3.** In this example the following system of integral–differential equations is solved, the exact solution are  $y_1(x) = x^2$ ,  $y_2(x) = (x + 1)^2$ .

$$\begin{cases} y_1' = x - y_1 + \int_0^x (y_2(t) - y_1(t))dt, & y_1(0) = 0; \\ y_2' = 3x + 3 - y_2 + \int_0^x (y_2(t) - y_1(t))dt, & y_2(0) = 1. \end{cases} \quad (21)$$

According to the variational iteration method we consider the correction functional in the following form (see [1, 3–6, 11–13]):

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \lambda_1(\tau) \{y_{1_n}' + y_{1_n} - \int_0^\tau (\tilde{y}_{2_n}(s) + \tilde{y}_{1_n}(s)) ds - \tau\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \lambda_2(\tau) \{y_{2_n}' + y_{2_n} + \int_0^\tau (\tilde{y}_{2_n}(s) + \tilde{y}_{1_n}(s)) ds - 3\tau - 3\} d\tau, \end{cases} \quad (22)$$

where  $\lambda_1(\tau)$  and  $\lambda_2(\tau)$  are the general Lagrange multipliers which its optimal value is found by using variational theory. the value of  $y_{1_0}$  and  $y_{2_0}$  are initial approximations and must be chosen suitably and  $\tilde{y}_{1_n}$  and  $\tilde{y}_{2_n}$  are the restricted values i.e.  $\delta \tilde{y}_{1_n} = 0$  and  $\delta \tilde{y}_{2_n} = 0$ . Now we have:

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau) \{y_{1_n}' + y_{1_n} - \int_0^\tau (\tilde{y}_{2_n}(s) + \tilde{y}_{1_n}(s)) ds - \tau\} d\tau, \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau) \{y_{2_n}' + y_{2_n} + \int_0^\tau (\tilde{y}_{2_n}(s) + \tilde{y}_{1_n}(s)) ds - 3\tau - 3\} d\tau, \end{cases} \quad (23)$$

to find the optimal value of  $\lambda_i$

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau) \{y_{1_n}' + y_{1_n}\} d\tau \\ \quad = \delta y_{1_n}(x) + \delta \lambda_1(\tau) y_{1_n}(\tau) |_{\tau=x} + \delta \int_0^x (\lambda_1(\tau) - \lambda_1'(\tau) y_{1_n}(\tau)) d\tau \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau) \{y_{2_n}' + y_{2_n}\} d\tau \\ \quad = \delta y_{2_n}(x) + \delta \lambda_2(\tau) y_{2_n}(\tau) |_{\tau=x} + \delta \int_0^x (\lambda_2(\tau) - \lambda_2'(\tau) y_{2_n}(\tau)) d\tau \end{cases} \quad (24)$$

and this yields to the following stationary conditions:

$$1 + \lambda_1(\tau) = 0|_{\tau=x}, \quad \lambda_1(\tau) - \lambda_1'(\tau) = 0|_{\tau=x}, \quad (25)$$

$$1 + \lambda_2(\tau) = 0|_{\tau=x}, \quad \lambda_2(\tau) - \lambda_2'(\tau) = 0|_{\tau=x}, \quad (26)$$

Therefore the Lagrange multipliers is defined as the following form:

$$\lambda_1(\tau) = x - \tau - 1, \quad \lambda_2(\tau) = x - \tau - 1 \quad (27)$$



substituting Eq. (27) into the correctional functional Eq. (23) we get the following iteration formula:

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x (x - \tau - 1) \{y'_{1_n} + y_{1_n} - \int_0^\tau (y_{2_n}(s) - y_{1_n}(s)) ds - \tau\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x (x - \tau - 1) \{y'_{2_n} + y_{2_n} - \int_0^\tau (y_{2_n}(s) - y_{1_{n+1}}(s)) ds - 3\tau - 3\} d\tau, \end{cases} \quad (28)$$

As say before, it is better to use below iteration formulas (28) instead of above iteration formulas (27) for obtaining better and closer approximations to the exact solution.

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x (x - \tau - 1) \{y'_{1_n} + y_{1_n} - \int_0^\tau (y_{2_n}(s) - y_{1_n}(s)) ds - \tau\} d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x (x - \tau - 1) \{y'_{2_n} + y_{2_n} + \int_0^\tau (y_{2_n}(s) - y_{1_{n+1}}(s)) ds - 3\tau - 3\} d\tau, \end{cases} \quad (28)$$

We start with initial approximations  $y_{1_0}(x) = y_1(0)$  and  $y_{2_0}(x) = y_2(0)$ , then by the iteration formulas (28) and MATLAB software we have

$$y_{1_1}(x) = x^2 + \left(-\frac{1}{3}x^3\right),$$

$$y_{2_1}(x) = x^2 + 2x + 1 + \left(-\frac{2}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{1}{360}x^6\right),$$

$$y_{1_2}(x) = x^2 + \left(-\frac{1}{30}x^5 + \frac{1}{840}x^7 - \frac{1}{6720}x^8 + \frac{1}{181440}x^9\right),$$

$$y_{2_2}(x) = x^2 + 2x + 1 + \left(-\frac{1}{15}x^5 + \frac{1}{360}x^7 - \frac{1}{3360}x^8 - \frac{1}{90720}x^9 + \frac{1}{302400}x^{10} + \dots\right),$$

$$y_{1_3}(x) = x^2 + \left(-\frac{1}{360}x^7 + \frac{1}{10080}x^8 + \frac{1}{25920}x^9 - \frac{1}{181440}x^{10} + \frac{1}{19958400}x^{11} + \dots\right),$$

$$y_{2_3}(x) = x^2 + 2x + 1 + \left(-\frac{1}{315}x^7 + \frac{1}{5040}x^8 + \frac{1}{10080}x^9 - \frac{1}{72576}x^{10} - \frac{1}{6652800}x^{11} + \dots\right),$$

⋮

$$y_{1_{13}}(x) = x^2 + \left(-\frac{1}{3786916514485104000000}x^{25} + \frac{1}{19691965875322540800000}x^{26} + \dots\right),$$

$$y_{1_{13}}(x) = x^2 + 2x + 1$$

$$+ \left(-\frac{1}{189345825742552000000}x^{25} + \frac{1}{984598299376661270400000}x^{26} + \dots\right),$$

⋮

and so on. Thus, we obtain

$$y_1(x) = \lim_{n \rightarrow \infty} y_{1_n}(x) = x^2,$$

$$y_2(x) = \lim_{n \rightarrow \infty} y_{2_n}(x) = x^2 + 2x + 1 = (x + 1)^2.$$

**Example 4** . In this example, we consider the following system of two nonlinear integral–differential equations, the exact solutions are  $y_1(x) = e^x$  and  $y_2(x) = e^{-x}$  .

$$\begin{cases} y_1' = y_1[x + 1 - \int_0^x y_1(s)y_2(s)ds], & y_1(0) = 1; \\ y_2' = y_2[x + 1 - \int_0^x y_1(s)y_2(s)ds], & y_2(0) = 1. \end{cases} \quad (29)$$

According to the variational iteration method we consider the correction functional in the following form (see [1, 3–6, 11–13]):

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) + \int_0^x \lambda_1(\tau)\{y_{1_n}' - \tilde{y}_{1_n}[\tau + 1 - \int_0^\tau \tilde{y}_{1_n}(s)\tilde{y}_{2_n}(s)ds]\}d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) + \int_0^x \lambda_2(\tau)\{y_{2_n}' - \tilde{y}_{2_n}[-\tau - 1 + \int_0^\tau \tilde{y}_{1_n}(s)\tilde{y}_{2_n}(s)ds]\}d\tau, \end{cases} \quad (30)$$

where  $\lambda_1(\tau)$  and  $\lambda_2(\tau)$  are the general Lagrange multipliers which its optimal value is found by using variational theory. the value of  $y_{1_0}$  and  $y_{2_0}$  are initial approximations and must be chosen suitably and  $\tilde{y}_{1_n}$  and  $\tilde{y}_{2_n}$  are the restricted values i.e.  $\delta\tilde{y}_{1_n} = 0$  and  $\delta\tilde{y}_{2_n} = 0$ . Now we have

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau)\{y_{1_n}' - \tilde{y}_{1_n}[\tau + 1 - \int_0^\tau \tilde{y}_{1_n}(s)\tilde{y}_{2_n}(s)ds]\}d\tau, \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau)\{y_{2_n}' - \tilde{y}_{2_n}[-\tau - 1 + \int_0^\tau \tilde{y}_{1_n}(s)\tilde{y}_{2_n}(s)ds]\}d\tau, \end{cases} \quad (31)$$

to find the optimal value of  $\lambda_i$

$$\begin{cases} \delta y_{1_{n+1}}(x) = \delta y_{1_n}(x) + \delta \int_0^x \lambda_1(\tau)\{y_{1_n}'\}d\tau = \delta y_{1_n}(x) + \delta \lambda_1(\tau)y_{1_n}(\tau) |_{\tau=x} - \delta \int_0^x \lambda_1'(\tau)y_{1_n}(\tau)d\tau, \\ \delta y_{2_{n+1}}(x) = \delta y_{2_n}(x) + \delta \int_0^x \lambda_2(\tau)\{y_{2_n}'\}d\tau = \delta y_{2_n}(x) + \delta \lambda_2(\tau)y_{2_n}(\tau) |_{\tau=x} - \delta \int_0^x \lambda_2'(\tau)y_{2_n}(\tau)d\tau, \end{cases} \quad (32)$$

and this yields to the following stationary conditions:

$$1 + \lambda_1(\tau) = 0|_{\tau=x}, \quad \lambda_1'(\tau) = 0|_{\tau=x}, \quad (33)$$

$$1 + \lambda_2(\tau) = 0|_{\tau=x}, \quad \lambda_2'(\tau) = 0|_{\tau=x}, \quad (34)$$

Therefore the Lagrange multipliers is defined as the following form:

$$\lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1. \quad (35)$$

Substituting Eq. (35) into the correctional functional Eq. (30) we get the following iteration formula:

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) - \int_0^x \{y_{1_n}' - y_{1_n}[\tau + 1 - \int_0^\tau y_{1_n}(s)y_{2_n}(s)ds]\}d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) - \int_0^x \{y_{2_n}' - y_{2_n}[-\tau - 1 + \int_0^\tau y_{1_n}(s)y_{2_n}(s)ds]\}d\tau, \end{cases} \quad (36)$$

As say before, it is better to use below iteration formulas (37) instead of above iteration formulas (36) for obtaining better and closer approximations to the exact solution

$$\begin{cases} y_{1_{n+1}}(x) = y_{1_n}(x) - \int_0^x \{y'_{1_n} - y_{1_n}[\tau + 1 - \int_0^\tau y_{1_n}(s)y_{2_n}(s)ds]\}d\tau, \\ y_{2_{n+1}}(x) = y_{2_n}(x) - \int_0^x \{y'_{2_n} - y_{2_n}[-\tau - 1 + \int_0^\tau y_{1_{n+1}}(s)y_{2_n}(s)ds]\}d\tau, \end{cases} \quad (37)$$

We start with initial approximations  $y_{1_0} = y_1(0)$  and  $y_{2_0}(x) = y_2(0)$ , then by the iteration formulas (37) and MATLAB software we have

$$\begin{aligned} y_{1_1}(x) &= 1 + x, \\ y_{2_1}(x) &= 1 - x + \left(\frac{1}{6}x^3\right), \\ y_{1_2}(x) &= 1 + x + \frac{1}{2}x^2 + \left(\frac{1}{12}x^4 + \frac{7}{120}x^5 - \frac{1}{80}x^6 - \frac{1}{120}x^7\right), \\ y_{2_2}(x) &= 1 - x + \frac{1}{2}x^2 + \left(-\frac{1}{12}x^4 + \frac{1}{60}x^5 + \frac{1}{45}x^6 - \frac{7}{720}x^7 - \frac{17}{40320}x^8 + \frac{853}{362880}x^9\right), \\ y_{1_3}(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \left(\frac{1}{60}x^5 + \frac{1}{720}x^6 - \frac{17}{2520}x^7 + \frac{5}{4032}x^8 + \frac{67}{120960}x^9 + \dots\right), \\ y_{2_3}(x) &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \left(\frac{1}{40}x^5 - \frac{7}{720}x^6 + \frac{1}{1680}x^7 + \frac{1}{5040}x^8 + \frac{47}{60480}x^9 + \dots\right), \\ y_{1_4}(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \left(\frac{1}{180}x^6 + \frac{1}{630}x^7 - \frac{1}{5760}x^8 + \frac{1}{5670}x^9 + \dots\right), \\ y_{2_4}(x) &= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \left(-\frac{1}{180}x^6 + \frac{11}{5040}x^7 - \frac{1}{2880}x^8 + \frac{1}{1209600}x^9 + \dots\right), \\ &\vdots \end{aligned}$$

and so on. Thus, we obtain

$$\begin{aligned} y_1(x) &= \lim_{n \rightarrow \infty} y_{1_n}(t) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \dots, \\ y_2(x) &= \lim_{n \rightarrow \infty} y_{2_n}(t) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 + \dots, \end{aligned}$$

which are quite close to the Taylor series of their exact solutions  $y_1(x) = e^x$  and  $y_2(x) = e^{-x}$ .

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