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Some fixed point theorems of self-generalized contractions in partially ordered G-metric spaces

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Abstract

The objective of this paper is to prove some fixed point results for self-mappings in partially ordered G-metric spaces using generalized contractive conditions. Our results are the extensions of the results presented in Agarwal et al. [R. P. Agarwal, M. A. El-Gebeily, D. O'Regan, Appl. Anal., **87** (2008), 109–116] form ordered metric spaces to partially ordered G-metric spaces. The usefulness of the results is also illustrated by an example.

Keywords: G-metric, partially ordered spaces, generalized contractions, fixed points.
2010 MSC: 47H10, 54H25.

1. Introduction

Banach fixed point theorem is of fundamental importance in fixed point theory. Banach fixed point theorem has been extended by many authors for different spaces. Recently Ran and Reuring [16] extended Banach fixed point theorem to partial ordered metric spaces. The results in [16] were improved in [15]. Some more fixed point results in partially ordered metric space can be found in [3].

Gahler introduced the concept of 2-metric spaces in [11] and [12]. Later Dhage introduced the concept of D-metric space and claimed it to be a generalization of 2-metric space and presented multiple results related to D-metric spaces in [6–10]. But Mustafa and Sims found that most of the claims made by Dhage were not correct and gave a generalized concept known as generalized metric space, briefly known as G-metric space. Mustafa and Sims in [14] discussed existence of fixed points in complete G-metric space. Following this paper, a number of authors established a number of fixed point theorems setting of G-metric space (see, e.g., [1, 2, 4] and [5]). Recently, Agarwal et al. in their paper [3] proved some fixed point theorems of generalized contractive mappings in partially ordered metric spaces. In this paper, we prove some fixed point results for self-mappings in partially ordered G-metric spaces using generalized contractive conditions. Our results are the extensions of the results presented in Agarwal et al. [3] from ordered metric spaces to partially ordered G-metric spaces.

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2. Preliminaries

Definition 2.1. Let $f : X \to X$ be a mapping. If $f(x) = x$ then "x" is said to be the fixed point of f.

Definition 2.2 ([13]). Let X be a non-empty set and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if $x = y = z$;

(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G_3) $G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);

(G_5) $G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2.3. Let there be a set X with "$\preceq$" as a binary relation on X. Then "$\preceq$" is called a partial order over X, if "$\preceq$" is reflexive, antisymmetric and transitive for all $x, y, z \in X$, i.e.,

(i) $x \preceq x$ (reflexivity);

(ii) if $x \preceq y$ and $y \preceq x$ then $x = y$ (antisymmetry);

(iii) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitivity).

Definition 2.4. A set with a partial order defined on it is called a partially ordered set, abbreviated as poset.

Definition 2.5. Let $(X, \preceq)$ be a poset with "$\preceq$" as a partial ordering on X. Suppose there is a G-metric defined on X. Then the triplet $(X, G, \preceq)$ is called a partially ordered G-metric space.

Definition 2.6 ([13]). Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, and therefore $\{x_n\}$ is G-convergent to x or $\{x_n\}$ G-converges to x. Thus, $x_n \to x$ in a G-metric space (X, G) if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geqslant k$.

Proposition 2.7 ([13]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) *(x_n) is G-convergent to x;*

(ii) *$G(x_n, x_n, x) \to 0$, as $n \to \infty$;*

(iii) *$G(x_n, x, x) \to 0$, as $n \to \infty$;*

(iv) *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

Definition 2.8 ([13]). Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy, if for every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geqslant k$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.9 ([13]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) *the sequence (x_n) is G-Cauchy;*

(ii) *for every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geqslant k$.*

Definition 2.10 ([13]). A G-metric space (X, G) is called G-complete, if every G-cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 2.11. Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$. Then F is called a non-decreasing map, if for $x, y \in X$, $x \preceq y$ then $F(x) \preceq F(y)$.

3. Fixed point theorems in partially ordered metric spaces

Followings are the results given by Agarwal et al. [3] for partially ordered metric spaces which will further be extended to partially ordered G-metric spaces.

Theorem 3.1 ([3]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume there is a non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ and also suppose $F : X \to X$ is a non-decreasing mapping with,*

$$d(F(x), F(y)) \leqslant \psi(d(x, y)), \quad \forall\, x \preceq y.$$

Also suppose either

(1) F *is continuous; or*

(2)

$$\begin{cases} \text{if } \{x_n\} \subseteq X \text{ is a non-decreasing sequence with } x_n \to x \text{ in } X \\ \text{then } x_n \preceq x, \text{ for all n holds.} \end{cases}$$

If there exists an $x_0 \in X$ with $x_0 \preceq F(x_0)$ then F has a fixed point.

Remark 3.2 ([3]). If $\psi : [0, \infty) \to [0, \infty)$ is a continuous function (or upper semi-continuous from the right) with $\psi(t) < t$ for $t > 0$ then $\lim_{n\to\infty} \psi^n(t) = 0$ for $t > 0$, since for fixed $t > 0$ if $a_n = \psi^n(t)$, then $a_n = \psi(a_{n-1}) \leqslant a_{n-1}$ so $a_n \downarrow \beta$ say, and $\beta = \psi(\beta)$ (or $\beta \leqslant \psi(\beta)$) so $\beta = 0$.

Theorem 3.3 ([3]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for each $t > 0$ and also suppose $F : X \to X$ is a non-decreasing mapping with,*

$$d(F(x), F(y)) \leqslant \psi(\max\{d(x, y), d(x, F(x)), d(y, F(y))\}), \quad \forall\, x \preceq y.$$

Also suppose either hypothesis (1) or (2) of Theorem 3.1 holds. If there exists a $x_0 \in X$ with $x_0 \preceq F(x_0)$ then F has a fixed point.

4. Fixed point theorems in partially ordered G-metric spaces

Now we extend the above-mentioned results to partially ordered G-metric spaces.

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set with a G-metric G defined on X such that (X, G) is G-complete. Assume there is a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$ and also suppose that $T : X \to X$ is a non-decreasing self-mapping with,*

$$G(T(x), T(y), T(z)) \leqslant \phi(G(x, y, z)), \quad \forall\, z \preceq y \preceq x. \tag{4.1}$$

Also suppose either

(a) T *is continuous; or*

(b) *if $\{x_n\} \subseteq X$ is a non-decreasing sequence with $x_n \to x$ in X then $x_n \preceq x$ for all n holds.*

If there exists an $x_0 \in X$ with $x_0 \preceq T(x_0)$ then T has a unique fixed point.

Proof. We claim that $\phi(t) < t$, for all $t > 0$, since if $t_0 \leqslant \phi(t_0)$ for $t_0 > 0$ then

$$t_0 \leqslant \phi(t_0) \leqslant \phi^2(t_0) \leqslant \cdots \leqslant \phi^n(t_0),$$

which in turn gives

$$t_0 \leqslant \phi^n(t_0),$$

for each $n = \{1, 2, \cdots\}$ which is a contradiction to the fact that $t_0 > 0$. Therefore $\phi(t) < t$. Also $\phi(0) = 0$.

Next, if $T(x_0) = x_0$ for some $x_0 \in X$, then x_0 will be a fixed point and we are done. So, suppose $T(x_0) \neq x_0$. Since $x_0 \preceq T(x_0)$ and T is non-decreasing map, therefore we have

$$x_0 \preceq T(x_0) \preceq T^2(x_0) \preceq \cdots \preceq T^n(x_0) \preceq T^{n+1}(x_0) \preceq \cdots .$$

Now, since $x_0 \preceq T(x_0) \preceq T^2(x_0)$ so by using (4.1)

$$G(T^3(x_0), T^2(x_0), T(x_0)) \leqslant \phi(G(T^2(x_0), T(x_0), x_0)),$$

and since $T^2(x_0) \preceq T^3(x_0)$, so by using (4.1)

$$\begin{aligned} G(T^4(x_0), T^3(x_0), T^2(x_0)) &\leqslant \phi(G(T^3(x_0), T^2(x_0), T(x_0)) \\ &\leqslant \phi^2(G(T^2(x_0), T(x_0), x_0)), \end{aligned}$$

then by the induction we have

$$G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \leqslant \phi^n(G(T^2(x_0), T(x_0), x_0)).$$

Let $\epsilon > 0$ be fixed. Choose $n \in \{1, 2, \cdots\}$ so that

$$G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \leqslant \epsilon - \phi(\epsilon), \tag{4.2}$$

as $\phi(\epsilon) < \epsilon$, so $\epsilon - \phi(\epsilon) > 0$ and $\epsilon - \phi(\epsilon) < \epsilon$. Now using $T^n(x_0) \preceq T^{n+1}(x_0)$, we have

$$\begin{aligned} G(T^{n+3}(x_0), T^{n+1}(x_0), T^n(x_0)) &\leqslant G(T^{n+3}(x_0), T^{n+2}(x_0), T^{n+2}(x_0)) \\ &\quad + G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \\ &\leqslant \phi(G(T^{n+2}(x_0), T^{n+1}(x_0), T^{n+1}(x_0)) \\ &\quad + [\epsilon - \phi(\epsilon)] \\ &\leqslant \phi(G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \\ &\quad + [\epsilon - \phi(\epsilon)] && \text{(by using (4.1) and (4.2))} \\ &\leqslant \phi[\epsilon - \phi(\epsilon)] + [\epsilon - \phi(\epsilon)] && \text{(by using (4.2))} \\ &\leqslant \phi(\epsilon) + [\epsilon - \phi(\epsilon)] \\ &\leqslant \epsilon. \end{aligned}$$

Therefore,

$$G(T^{n+3}(x_0), T^{n+1}(x_0), T^n(x_0)) \leqslant \epsilon. \tag{4.3}$$

Also,

$$\begin{aligned} G(T^{n+4}(x_0), T^{n+1}(x_0), T^n(x_0)) &\leqslant G(T^{n+4}(x_0), T^{n+2}(x_0), T^{n+2}(x_0)) \\ &\quad + G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \\ &\leqslant (G(T^{n+4}(x_0), T^{n+2}(x_0), T^{n+2}(x_0)) \\ &\quad + [\epsilon - \phi(\epsilon)] && \text{(by using (4.2))} \\ &\leqslant (G(T^{n+4}(x_0), T^{n+2}(x_0), T^{n+1}(x_0)) \\ &\quad + [\epsilon - \phi(\epsilon)] \\ &\leqslant \phi(G(T^{n+3}(x_0), T^{n+1}(x_0), T^n(x_0)) && \text{(by using (4.1))} \\ &\quad + [\epsilon - \phi(\epsilon)] \\ &\leqslant \phi(\epsilon) + [\epsilon - \phi(\epsilon)] && \text{(by using (4.3))} \\ &\leqslant \epsilon. \end{aligned}$$

So we have,

$$G(T^{n+4}(x_0), T^{n+1}(x_0), T^n(x_0)) \leqslant \epsilon.$$

This gives by the induction,

$$G(T^{n+k}(x_0), T^{n+1}(x_0), T^n(x_0)) \leqslant \epsilon, \quad \text{for } k \in \{2, 3, \cdots\}.$$

Hence $T^n(x_0)$ is a G-cauchy sequence in X. Also since X is G-complete so there exists an $x \in X$ with $\lim_{n\to\infty} T^n(x_0) = x$.

Now if T is continuous then $T(x) = x$ holds since,

$$T(x) = \lim_{n\to\infty} T(T^{n-1}(x_0)) = \lim_{n\to\infty} T^n(x_0) = x.$$

Next suppose that $\{x_n\} \subseteq X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \preceq x$ for all n.

Suppose, $G(x, T(x), T(x)) = a > 0$, i.e., $x \neq T(x)$.

Since $\lim_{n\to\infty} T^n(x_0) = x$, so there exists $N \in \{1, 2, \cdots\}$ such that

$$G(T^n(x_0), x, x) < \frac{a}{2}, \quad \forall\, n \geqslant N. \tag{4.4}$$

Then for $n \geqslant N$, we have

$$\begin{aligned} G(x, T(x), T(x)) &\leqslant G(x, T^{n+1}(x_0), T^{n+1}(x_0)) + G(T^{n+1}(x_0), T(x), T(x)) \\ &< \frac{a}{2} + \phi(G(T^n(x_0), x, x)) && \text{(by using (4.4) and (4.1))} \\ &< \frac{a}{2} + \phi(\frac{a}{2}) && \text{(by using (4.4))} \\ &< \frac{a}{2} + \frac{a}{2} = a. \end{aligned}$$

This gives,

$$G(x, T(x), T(x)) < a,$$

which is a contradiction. Therefore, $T(x) = x$. Hence in both cases T has a fixed point.

Now to show that fixed point of T is unique, consider w be another fixed point of T. Then $T(w) = w, x \neq w$. Then,

$$G(x, x, w) = G(T(x), T(x), T(w)) \leqslant \phi(G(x, x, w)) < G(x, x, w),$$

which is a contradiction. Therefore, fixed point of T is unique. □

Example 4.2. Let $X = \mathbb{R}$ and $G : X \times X \times X \to \mathbb{R}_+$. Define

$$G(x, y, z) = |x - y| + |y - z| + |x - z|,$$

for all $x, y, z \in X$. Then X is a complete G-metric space. Also $x \preceq y$, if $y \leqslant x$ is partial ordering on X.

Let $T(x) = \frac{x}{10}$ and $\phi(t) = \frac{9}{10}t$. Clearly, $\phi^n(t) = \left(\frac{9}{10}\right)^n t \to 0$ as $n \to \infty$. Then

$$G(T(x), T(y), T(z)) = G(\frac{x}{10}, \frac{y}{10}, \frac{z}{10}) = \frac{1}{10}[|x - y| + |y - z| + |x - z|],$$

and

$$\phi(G(x, y, z)) = \phi(|x - y| + |y - z| + |x - z|) = \frac{9}{10}[|x - y| + |y - z| + |x - z|].$$

Therefore,

$$G(T(x), T(y), T(z)) < \phi(G(x, y, z)).$$

Also T has a fixed point at $x = 0$, since $T(0) = \frac{0}{10} = 0$. And this fixed point is unique as well.

Theorem 4.3. *Let* $(X, \preceq)$ *be a partially ordered set, let* G *be a G-metric on* X *such that* (X, G) *is a complete G-metric space. Assume there is a continuous function* $\phi : [0, \infty) \to [0, \infty)$ *with* $\phi(t) < t$ *for each* $t > 0$ *and also suppose that* $T : X \to X$ *is a non-decreasing mapping with,*

$$G(T(x), T(y), T(z)) \leqslant \phi(\max\{G(x, y, z), G(x, T(x), T(x)), G(y, T(y), T(y)), \\ G(z, T(z), T(z))\}), \quad \forall\, z \preceq y \preceq x.$$

Also suppose either

(a) T *is continuous; or*

(b) *if* $\{x_n\} \subseteq X$ *is a non-decreasing sequence with* $x_n \to x$ *in* X *then* $x_n \preceq x$ *for all* n *holds.*

If there exists an $x_0 \in X$ *with* $x_0 \preceq T(x_0)$*, then* T *has a unique fixed point.*

Proof. By Remark 3.2,

$$\lim_{n\to\infty} \phi^n(t) = 0, \quad \forall\, t > 0.$$

Let

$$\lambda_n = G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)).$$

Since, $T^{n-1}(x_0) \preceq T^n(x_0)$, so

$$\begin{aligned}
\lambda_n &\leqslant \phi(\max\{G(T^{n+1}(x_0), T^n(x_0), T^{n-1}(x_0)), G(T^{n+1}(x_0), T^{n+2}(x_0), T^{n+2}(x_0)), \\
&\quad G(T^n(x_0), T^{n+1}(x_0), T^{n+1}(x_0)), G(T^{n-1}(x_0), T^n(x_0), T^n(x_0))\}) \\
&\leqslant \phi(\max\{G(T^{n+1}(x_0), T^n(x_0), T^{n-1}(x_0)), G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)), \\
&\quad G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)), G(T^{n+1}(x_0), T^n(x_0), T^{n-1}(x_0))\}) \qquad \text{(by } (G_3)) \\
&\leqslant \phi(\max\{G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)), G(T^{n+1}(x_0), T^n(x_0), T^{n-1}(x_0))\}) \\
&= \phi(\max\{\lambda_n, \lambda_{n-1}\}).
\end{aligned}$$

Now we show that,

$$\lambda_n \leqslant \phi(\lambda_{n-1}). \tag{4.5}$$

If $\max(\lambda_n, \lambda_{n-1}) = \lambda_{n-1}$, then Equation (4.5) holds.

If $\max(\lambda_n, \lambda_{n-1}) = \lambda_n$, then

$$\lambda_n \leqslant \phi(\lambda_n).$$

Therefore, by the remark, $\lambda_n = 0$ and hence Equation (4.5) is satisfied.

Now since,

$$\lambda_n \leqslant \phi(\lambda_{n-1}) \leqslant \lambda_{n-1},$$

since $\phi(t) < t$, so there exists $\lambda \geqslant 0$ with $\lambda_n \downarrow \lambda$.

Since $\lambda_n \leqslant \phi(\lambda_{n-1})$ and ϕ is continuous, so

$$\lambda \leqslant \phi(\lambda).$$

This gives, $\lambda = 0$, and therefore

$$\lambda_n = G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)) \to 0, \quad \text{as } n \to \infty.$$

We claim $\{T^n(x_0)\}$ is a Cauchy sequence.

Suppose this does not hold. Then we can find a $\delta > 0$ and three sequences of integers

$$\{n(k), s(k), t(k)\}, \quad n(k) > s(k) > t(k) \geqslant k,$$

with

$$m_k = G(T^{n(k)}(x_0), T^{s(k)}(x_0), T^{t(k)}(x_0)) \geqslant \delta, \quad \text{for} \quad k \in \{1, 2, \cdots\}. \tag{4.6}$$

Also assume that,

$$G(T^{n(k)-1}(x_0), T^{s(k)-1}(x_0), T^{t(k)}(x_0)) < \delta,$$

by choosing $n(k)$ and $s(k)$ to be the smallest numbers exceeding $t(k)$ for which (4.6) holds.

Now, we show that m_k converges to δ as follows,

$$\begin{aligned}
\delta \leqslant m_k &= G(T^{n(k)}(x_0), T^{s(k)}(x_0), T^{t(k)}(x_0)) \\
&\leqslant G(T^{n(k)}(x_0), T^{n(k)-1}(x_0), T^{n(k)-1}(x_0)) \\
&\quad + G(T^{n(k)-1}(x_0), T^{s(k)}(x_0), T^{t(k)}(x_0)) \\
&\leqslant G(T^{n(k)}(x_0), T^{n(k)-1}(x_0), T^{n(k)-2}(x_0)) \\
&\quad + G(T^{n(k)-1}(x_0), T^{s(k)}(x_0), T^{t(k)}(x_0)) \\
&\leqslant \lambda_{n(k)-2} + G(T^{n(k)-1}(x_0), T^{s(k)}(x_0), T^{t(k)}(x_0)) \\
&\leqslant \lambda_{n(k)-2} + G(T^{s(k)}(x_0), T^{n(k)-1}(x_0), T^{t(k)}(x_0)) \\
&\leqslant \lambda_{n(k)-2} + G(T^{s(k)}(x_0), T^{s(k)-1}(x_0), T^{s(k)-1}(x_0)) \\
&\quad + G(T^{s(k)-1}(x_0), T^{n(k)-1}(x_0), T^{t(k)}(x_0)) \\
&< \lambda_{n(k)-2} + G(T^{s(k)}(x_0), T^{s(k)-1}(x_0), T^{s(k)-2}(x_0)) + \delta \\
&< \lambda_{n(k)-2} + \lambda_{s(k)-2} + \delta.
\end{aligned}$$

This implies,

$$\delta \leqslant \lim_{n \to \infty} m_k < \delta,$$

that is $\delta < \delta$, a contradiction.

Hence $T^n(x_0)$ is a Cauchy sequence. As X is complete G-metric so, there exists an $x \in X$ such that $\lim_{n\to\infty} T^n(x_0) = x$. Further, if (a) is true then clearly $T(x) = x$.

If (b) holds, then

$$\begin{aligned}
G(x, T(x), T(x)) &\leqslant G(x, T^{n+1}(x_0), T^{n+1}(x_0)) + G(T^{n+1}(x_0), T(x), T(x)) \\
&\leqslant G(x, T^{n+1}(x_0), T^{n+1}(x_0)) + \phi(\max\{G(T^n(x_0), x, x), \\
&\quad G(T^n(x_0), T^{n+1}(x_0), T^{n+1}(x_0)), \\
&\quad G(x, T(x), T(x)), G(x, T(x), T(x))\}) \\
&\leqslant G(x, T^{n+1}(x_0), T^{n+1}(x_0)) + \phi(\max\{G(T^n(x_0), x, x), \\
&\quad G(T^{n+2}(x_0), T^{n+1}(x_0), T^n(x_0)), G(x, T(x), T(x))\}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(x, T(x), T(x)) \leqslant G(x, T^{n+1}(x_0), T^{n+1}(x_0)) + \phi(\max\{G(T^n(x_0), x, x), \\
\lambda_n, G(x, T(x), T(x))\}).
\end{aligned}$$

Since ϕ is continuous so as $n \to \infty$, we get,

$$G(x, T(x), T(x)) \leqslant \phi(G(x, T(x), T(x))).$$

Therefore,

$$G(x, T(x), T(x)) = 0,$$

which in turn gives,

$$T(x) = x.$$

Now for uniqueness, let w be another fixed point. Then, $T(w) = w$. Therefore,

$$\begin{aligned} G(x,x,w) &= G(T(x),T(x),T(w)) \\ &\leqslant \phi(\max\{G(x,x,w),G(x,T(x),T(x)), \\ &\quad G(q,T(x),T(x)),G(w,T(w),T(w))\}) \\ &\leqslant \phi(\max\{G(x,x,w),G(x,T(x),T(x)),G(w,T(w),T(w))\}) \\ &\leqslant \phi(\max\{G(x,x,w),G(x,x,x),G(w,w,w)\}) \\ &\leqslant \phi(\max\{G(x,x,w),0,0\}). \end{aligned}$$

This gives,

$$G(x,x,w) \leqslant \phi(G(x,x,w)).$$

So by Remark 3.2,

$$G(x,x,w) = 0,$$

which finally implies,

$$x = w.$$

Hence, fixed point is unique. □

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