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Asymptotic behavior criteria for solutions of nonlinear third order neutral differential equations

Hussain Ali Mohamad[a,*], Sattar Naser Ketab[b]

[a] *Department of mathematics, College of science for women, University of Baghdad, IRAQ.*
[b] *Department of Mathematics, College of education, University of Kufa, IRAQ.*

Abstract

In this paper the asymptotic behavior for all nonoscillatory solutions of third order nonlinear neutral differential equations have been investigated, where some necessary and sufficient conditions are obtained to guarantee the convergence of these solutions to zero or tends to infinity as $t \to \infty$. We introduced Lemma 2.1 and Lemma 2.2 which are a generalization of Lemma 1.5.2 [I. Győri, G. Ladas, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, (1991)]. Some examples are given to illustrate our main results.

Keywords: Oscillation, asymptotic behavior, neutral differential equations.
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1. Introduction

The oscillation theory and the asymptotic behavior criteria for neutral differential equations NDE received attention of many authors in the last several years for their widely achieve in many applications. By a solution of (1.1), we mean a function $y \in C([t_y, \infty); \mathbb{R})$, $t_y \geqslant t_0$, which has the property $y(t) + p(t)f(y(\tau(t))) \in C^3([t_y, \infty), \mathbb{R})$, and satisfies (1.1) on $[t_y, \infty)$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_y, \infty)$ otherwise, it is called nonoscillatory, that is a solution is called nonoscillatory if it is eventually positive or eventually negative. Jaros and Kusano [9] established sufficient conditions under which all proper solutions of higher order linear NDE are oscillatory where $F(t) \equiv 0$. Gyori and Ladas [8], Das et al. [3], obtained sufficient conditions for higher order NDE with constant and variable delays. Parhi et al. [17], and Rath et al. [18] obtained sufficient conditions for all solutions of (1.1) to oscillate or tend to zero as $t \to \infty$, where the delays are constants and $f(y) = y$. Mohamad [15], Mohamad and Ketab [16] obtained sufficient conditions for oscillation of all solutions of the linear third order NDE. Karpuz et al. [12] compared oscillatory and asymptotic behaviors of all solutions of higher-order linear NDE with first-order delay differential equations, depending on two different ranges of the coefficient associated with the neutral part. El-Sheikh et al. [6] studied the oscillatory behavior of solutions of general third order NDEs $[a(t)(b(t)z'(t))']' + f(t, z(t), z'(t)) = 0$, where $z(t) = x(t) + p(t)x(\tau(t))$ using

*Corresponding author
Email address: hussainmohamad22@gmail.com (Hussain Ali Mohamad)

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a generalized Riccati transformation. Jiang and Li [11] studied asymptotic nature of a class of third-order NDEs $(r(t)[x(t)+P(t)x(t-\tau(t))]'')'+\sum_{i=1}^{m} Q_i(t)f_i(x(t-\sigma_i(t)))=0,\ t \geqslant t_0$ by using a generalized Riccati substitution and the integral averaging technique, a new Philos-type criterion is obtained which ensures that every solution of the studied equation is either oscillatory or converges to zero. In this paper we study asymptotic behavior of (1.1) and established some necessary and sufficient conditions to insure the convergence of all solutions of (1.1) to zero or tend to infinity. Some examples are given to illustrate the obtained results.

$$\frac{d^3}{dt^3}\left[y(t)+p(t)\,f(y(\tau(t)))\right]+q(t)\,g(y(\sigma(t)))=F(t), \quad t \geqslant t_0. \tag{1.1}$$

Under the following assumptions:

(A_1) $p(t) \in C([t_0,\infty);\mathbb{R}^+),\ q(t) \in C([t_0,\infty);\mathbb{R})$.

(A_2) $\tau(t),\ \sigma(t) \in C([t_0,\infty);\mathbb{R}),\ \lim_{t\to\infty}\tau(t)=\infty,\ \lim_{t\to\infty}\sigma(t)=\infty$, where $\tau(t), \sigma(t)$ are increasing functions.

(A_3) $f \in C(\mathbb{R};\mathbb{R}),\ \frac{f(u)}{u} \leqslant \delta_1,\ u\,f(u)>0,\ \delta_1>0$ is constant.

(A_4) There exists a function $h(t) \in C^3([t_0,\infty);\mathbb{R})$, such that $\lim_{t\to\infty} h(t)=0$ and $h'''(t)=F(t)$.

(A_5) $g \in C(\mathbb{R};\mathbb{R}),\ \frac{g(u)}{u} \geqslant \beta>0$.

(A_3') $f \in C(\mathbb{R};\mathbb{R}),\ \delta_2 \leqslant \frac{f(u)}{u} \leqslant \delta_1,\ \delta_1,\delta_2>0$.

2. Asymptotic behavior of Equation (1.1)

In this section, we obtain some main results, for simplicity define the function

$$z(t)=y(t)+p(t)f(y(\tau(t)))-h(t). \tag{2.1}$$

Using (2.1) into (1.1) leads to

$$z'''(t)=-q(t)g(y(\sigma(t))). \tag{2.2}$$

The following lemmas generalized [8, lemma 1.5.2]:

Lemma 2.1. *Let* $u,\ x,\ \gamma,\ \tau,\ h:[t_0,\infty)\to\mathbb{R}$, *be such that*

$$u(t)=x(t)+\gamma(t)f(x(\tau(t)))-h(t), \tag{2.3}$$

where $\lim_{t\to\infty} h(t)=0,\ \tau(t)$ *is strictly increasing,* $\lim_{t\to\infty}\tau(t)=\infty$. *Assume that* ($A_3$) *holds,* $0 \leqslant \gamma(t) \leqslant \gamma_1 < \frac{1}{\delta_1}$. *If* $x(t)$ *is an eventually positive (or eventually negative), such that* $\liminf_{t\to\infty} x(t)=0$ *and* $\lim_{t\to\infty} u(t)=L \in R$ *exists. Then* $\lim_{t\to\infty} u(t)=\lim_{t\to\infty} x(t)=0$.

Proof. Let $x(t)>0,\ x(\tau(t)>0$ for $t \geqslant t_0$, then from (2.3) we get

$$\begin{aligned} u(\tau(t)) &= x(\tau(t))+\gamma(\tau(t))f(x(\tau(\tau(t))))-h(\tau(t)),\\ u(t)-u(\tau(t)) &= x(t)+\gamma(t)f(x(\tau(t)))-h(t)-x(\tau(t))-\gamma(\tau(t))f(x(\tau(\tau(t))))+h(\tau(t))\\ &\leqslant x(t)+\gamma_1\,\delta_1\,x(\tau(t))-h(t)-x(\tau(t))+h(\tau(t)). \end{aligned}$$

Since $\liminf_{t\to\infty} x(t)=0$, let t_n be a sequence of points such that $\lim_{n\to\infty} t_n=\infty$ and $\lim_{n\to\infty} x(t_n)=0$. Then

$$u(t_n)-u(\tau(t_n)) \leqslant x(t_n)+\gamma_1\,\delta_1\,x(\tau(t_n))-x(\tau(t_n))-h(t_n)+h(\tau(t_n)),$$

as $n \to \infty$ the last inequality leads to:

$$\lim_{n\to\infty} u(t_n) - \lim_{n\to\infty} u(\tau(t_n)) \leqslant \lim_{n\to\infty} x(t_n) + (\gamma_1\delta_1 - 1) \lim_{n\to\infty} x(\tau(t_n)) - \lim_{n\to\infty} h(t_n) + \lim_{n\to\infty} h(\tau(t_n)),$$

$$0 \leqslant \left(\gamma_1\delta_1 - 1\right) \lim_{n\to\infty} x(\tau(t_n)),$$

which implies that $\lim_{n\to\infty} x(\tau(t_n)) = 0$.

From (2.3) we get

$$u(t) \geqslant x(t) - h(t),$$
$$u(t_n) \geqslant x(t_n) - h(t_n),$$

as $n \to \infty$, it follows that $L \geqslant 0$. On the other hand from (2.3) we have

$$u(t) \leqslant x(t) + \gamma_1\delta_1\, x(\tau(t)) - h(t),$$

$$u(t_n) \leqslant x(t_n) + \gamma_1\delta_1\, x(\tau(t_n)) - h(t_n),$$

as $n \to \infty$, it follows that $L \leqslant 0$, hence $L = 0$. Finally

$$0 < x(t) \leqslant u(t) + h(t),$$

so from the last inequality we conclude that $\lim_{t\to\infty} x(t) = 0$. □

Lemma 2.2. *Let* $u, x, \gamma, \tau, h : [t_0, \infty) \to \mathbb{R}$, *satisfy* (2.3) *where* $\lim_{t\to\infty} h(t) = 0$, *and let* $\tau(t)$ *be strictly increasing,* $\lim_{t\to\infty} \tau(t) = \infty$.

Assume that (A_3') *holds,* $\frac{1}{\delta_2} < \gamma_2 \leqslant \gamma(t) \leqslant \gamma_1$. *If* $x(t)$ *is an eventually positive (or eventually negative), such that* $\liminf_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} u(t) = L \in R$ *exists. Then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t) > 0$, $x(\tau(t) > 0$, for $t \geqslant t_0$, then from (2.3) we get

$$u(\tau^{-1}(t)) = x(\tau^{-1}(t)) + \gamma(\tau^{-1}(t))f(x(t)) - h(\tau^{-1}(t)),$$

$$\begin{aligned} u(\tau^{-1}(t)) - u(t) &= x(\tau^{-1}(t)) + \gamma(\tau^{-1}(t))f(x(t)) - h(\tau^{-1}(t)) - x(t) \\ &\quad - \gamma(t)f(x(\tau(t)) + h(t) \\ &\geqslant (\gamma_2\delta_2 - 1)x(t) - \gamma_1\delta_1 x(\tau(t)) - h(\tau^{-1}(t)) + h(t). \end{aligned}$$

Since $\liminf_{t\to\infty} x(t) = 0$, let $\tau(t_n)$ be a sequence of points such that $\lim_{n\to\infty} \tau(t_n) = \infty$ and

$$\lim_{n\to\infty} x(\tau(t_n)) = 0.$$

Then

$$\begin{aligned} u(\tau^{-1}(t_n)) - u(t_n) &= x(\tau^{-1}(t_n)) + \gamma(\tau^{-1}(t_n))f(x(t_n)) - h(\tau^{-1}(t_n)) - x(t_n) \\ &\quad - \gamma(t_n)f(x(\tau(t_n)) + h(t_n), \end{aligned}$$

as $n \to \infty$ the last inequality leads to:

$$\lim_{n\to\infty} u(\tau^{-1}(t_n)) - \lim_{n\to\infty} u(t_n) \geqslant (\gamma_2\delta_2 - 1) \lim_{n\to\infty} x(t_n) - \gamma_1\delta_1 \lim_{n\to\infty} x(\tau(t_n)) - \lim_{n\to\infty} h(\tau^{-1}(t_n)) + \lim_{n\to\infty} h(t_n),$$

$$0 \geqslant \left(\gamma_2\delta_2 - 1\right) \lim_{n\to\infty} x(t_n),$$

which leads to $\lim_{n\to\infty} x(t_n) = 0$. From (2.3) we get

$$u(t) \geqslant x(t) - h(t),$$

$$u(t_n) \geqslant x(t_n) - h(t_n),$$

as $n \to \infty$, it follows that $L \geqslant 0$. On the other hand from (2.3) we get

$$u(t) \leqslant x(t) + \gamma_1 \delta_1 x(\tau(t)) - h(t),$$

$$u(t_n) \leqslant x(t_n) + \gamma_1 \delta_1 x(\tau(t_n)) - h(t_n),$$

as $n \to \infty$, it follows that $L \leqslant 0$, hence $L = 0$. Finally

$$0 < x(t) \leqslant u(t) + h(t),$$

so from the last inequality we conclude that $\lim_{t\to\infty} x(t) = 0$. □

Theorem 2.3. *Assume that* A_1–A_5 *hold,* $0 \leqslant p(t) < \frac{1}{\delta_1}$, $q(t) \geqslant 0$, *in addition to*

$$\liminf_{t\to\infty} q(t) > 0. \tag{2.4}$$

Then every nonoscillatory solution of (1.1) *tends to zero as* $t \to \infty$.

Proof. Assume for the sake of contradiction that (1.1) has a nonoscillatory solution, let $y(t)$ be eventually positive solution of (1.1) (the case where $y(t)$ be eventually negative is similar and will be omitted). Let $y(t) > 0$, $y(\tau(t)) > 0$, $y(\sigma(t)) > 0$, for $t \geqslant t_0$. From (2.2) it follows that

$$z'''(t) \leqslant 0, \quad t \geqslant t_0,$$

which implies that $z''(t)$, $z'(t)$, $z(t)$, are monotone functions. We have two cases for $z''(t)$:

Case 1. $z''(t) < 0$, $t \geqslant t_1 \geqslant t_0$, thus $z'(t) < 0, z(t) < 0$ and $\lim_{t\to\infty} z(t) = -\infty$.

From (2.1) we get $z(t) \geqslant -h(t)$, thus $\lim_{t\to\infty} h(t) = \infty$, which is a contradiction.

Case 2. $z''(t) > 0$, $t \geqslant t_1 \geqslant t_0$. We have two cases for $z'(t)$:

Case 2.1. $z'(t) > 0$, $t \geqslant t_2 \geqslant t_1$. $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leqslant 0$, and $\lim_{t\to\infty} z(t) = \infty$. We claim that $\lim_{t\to\infty} y(t) = \infty$, otherwise there exists $k > 0$ such that $y(t) \leqslant k$, and from (2.1) with virtue of $\lim_{t\to\infty} h(t) = 0$, we get $z(t) < k + k - h(t)$ which implies that $\lim_{t\to\infty} z(t) < \infty$ a contradiction.

Condition (2.4) implies that there exist $q_* > 0$ and $t_3 \geqslant t_2$ such that $q(t) \geqslant q_*$ for $t \geqslant t_3$. From (2.2) and (A_5) we get

$$z'''(t) \leqslant -\beta\, q(t)\, y(\sigma(t)). \tag{2.5}$$

Integrating (2.5) from t_3 to t we get

$$z''(t) - z''(t_3) \leqslant -\beta \int_{t_3}^{t} q(s) y(\sigma(s))\, ds,$$

$$z''(t) - z''(t_3) \leqslant -\beta\, q_* \int_{t_3}^{t} y(\sigma(s))\, ds,$$

$$\int_{t_3}^{t} y(\sigma(s))\, ds \leqslant \frac{z''(t_3) - z''(t)}{\beta\, q_*} < \infty.$$

The last inequality leads to a contradiction since $\lim_{t\to\infty} \int_{t_3}^{t} y(\sigma(s)) ds = \infty$.

Case 2.2. $z'(t) < 0$, $t \geqslant t_2 \geqslant t_1$. We have two cases for $z(t)$:

Case 2.2.1. $z(t) < 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) \leqslant 0$, $t \geqslant t_3 \geqslant t_2$.

Let $\lim_{t\to\infty} z(t) = L$, $\quad -\infty \leqslant L < 0$.

If $L = -\infty$, we can use the same treatment in Case 1 to show that $\lim_{t\to\infty} h(t) = \infty$, which is a contradiction.

If $-\infty < L < 0$, then there exist $\alpha > 0$ and $t_* \geqslant t_3$ such that $z(t) \leqslant -\alpha$, $t \geqslant t_*$, from (2.1) we get

$$y(t) = z(t) - p(t) f(y(\tau(t))) + h(t) \leqslant -\alpha + h(t),$$

hence $y(t) < -\alpha + \varepsilon,\ \varepsilon > 0$, then $y(t) \leqslant -\alpha$ which is a contradiction.

Case 2.2.2. $z(t) > 0,\ z'(t) < 0,\ z''(t) > 0,\ z'''(t) \leqslant 0,\ t \geqslant t_3 \geqslant t_2$.

We claim that $\liminf_{t\to\infty} y(t) = 0$, otherwise $\liminf_{t\to\infty} y(t) > 0$, then there exist $c > 0$ and $t_4 \geqslant t_3$ such that $y(t) \geqslant c,\ t \geqslant t_4$, from (2.2) we get

$$z'''(t) \leqslant -q_*\,\beta\, y(\sigma(t)). \tag{2.6}$$

Integrating (2.6) from t_4 to t we get

$$z''(t) - z''(t_4) \leqslant -q_*\beta \int_{t_4}^{t} y(\sigma(s))\,ds,$$

$$z''(t) - z''(t_4) \leqslant -q_*\beta\, c\,(t - t_4),$$

which implies that $\lim_{t\to\infty} z''(t) = -\infty$, which is a contradiction. Hence $\liminf_{t\to\infty} y(t) = 0$, so by Lemma 2.1 it follows that $\lim_{t\to\infty} y(t) = 0$. □

Example 2.4. Consider the neutral differential equation:

$$[y(t) + (2 + \frac{1}{t})f(y(\frac{t}{2}))]''' + (1 + \frac{1}{t})g(y(t)) = \frac{36t + 48}{t^6} + \frac{1}{t} + \frac{1}{t^2}, \quad t \geqslant 1, \tag{2.7}$$

$\tau(t) = \frac{t}{2},\ \sigma(t) = t,\ p(t) = 2 + \frac{1}{t},\ f(y(\frac{t}{2})) = 1 + y(\frac{t}{2}),\ q(t) = 1 + \frac{1}{t},\ F(t) = \frac{36t+48}{t^6} + \frac{1}{t} + \frac{1}{t^2},\ g(y(t)) = \frac{36t+48}{t^5} + y(t)$. One can find that all conditions of Theorem 2.3 are held. To see the condition (2.4):

$$\liminf_{t\to\infty} q(t) = \lim_{t\to\infty} (1 + \frac{1}{t}) = 1 > 0.$$

So, every nonoscillatory solution of (2.7) tends to zero as $t \to \infty$. For instance $y(t) = \frac{1}{t}$ is such a solution.

Theorem 2.5. *Assume that* A_1–A_5 *hold,* $0 \leqslant p(t) \leqslant p* < \frac{1}{\delta},\ q(t) \leqslant 0$ *addition to*

$$\limsup_{t\to\infty} q(t) < 0. \tag{2.8}$$

Then every nonoscillatory solution $y(t)$ *of* (1.1) *tends to zero or* $|y(t)| \to \infty$ *as* $t \to \infty$.

Proof. Let $y(t)$ be an eventually positive solution of (1.1). Thus for $t \geqslant t_0,\ y(t) > 0,\ y(\tau(t)) > 0,\ y(\sigma(t)) > 0$. From (2.2) it follows that $z'''(t) \geqslant 0,\ t \geqslant t_0$, which implies that $z''(t),\ z'(t),\ z(t)$ are monotone functions. We have two cases for $z''(t)$:

Case 1. $z''(t) > 0,\ t \geqslant t_1 \geqslant t_0$, hence $z'(t) > 0,\ z(t) > 0$ and $\lim_{t\to\infty} z(t) = \infty$, which implies that $\lim_{t\to\infty} y(t) = \infty$, otherwise $\lim_{t\to\infty} y(t) < \infty$, so there exists $c > 0$ such that $y(t) \leqslant c,\ t \geqslant t_* \geqslant t_1$ then from (2.1) we get $z(t) \leqslant c + p^*\,\delta\, c - h(t)$ implies that $\lim_{t\to\infty} z(t) < \infty$, which is a contradiction.

Case 2. $z''(t) < 0,\ t \geqslant t_1 \geqslant t_0$. We have two cases for $z'(t)$:

Case 2.1. $z'(t) < 0, t \geqslant t_2 \geqslant t_1$, in this case we have $z(t) < 0,\ z'(t) < 0,\ z''(t) < 0, z'''(t) \geqslant 0$ and $\lim_{t\to\infty} z(t) = -\infty$.

From (2.1) we get $z(t) > -h(t)$, it follows that $\lim_{t\to\infty} h(t) = \infty$, which is a contradiction.

Case 2.2. $z'(t) > 0, t \geqslant t_2 \geqslant t_1$. We have two cases for $z(t)$:

Case 2.2.1. $z(t) > 0, t \geqslant t_3 \geqslant t_2$, thus $z(t) > 0,\ z'(t) > 0,\ z''(t) < 0,\ z'''(t) \geqslant 0$.

Let $\lim_{t\to\infty} z(t) = L,\ 0 < L \leqslant \infty$. If $L = \infty$ we can use the same treatment in Case 1 to show that

$\lim_{t\to\infty} y(t) = \infty$. If $0 < L < \infty$, then by (2.8) there exists $q^* < 0$ such that $q(t) \leqslant q^* < 0$. From (2.2) and (A_5) we get

$$z'''(t) \geqslant -\beta\, q(t)\, y(\sigma(t)). \tag{2.9}$$

Integrating (2.9) from t_3 to t we get

$$z''(t) - z''(t_3) \geqslant -\beta \int_{t_3}^{t} q(s)\, y(\sigma(s))\, ds,$$

$$z''(t) - z''(t_3) \geqslant -\beta q^* \int_{t_3}^{t} y(\sigma(s))\, ds,$$

$$\int_{t_3}^{t} y(\sigma(s))\, ds \leqslant \frac{z''(t) - z''(t_3)}{-\beta q^*} < \infty, \quad t \geqslant t_3.$$

The last inequality implies that $\liminf_{t\to\infty} y(t) = 0$. By Lemma 2.1, it follows that $\lim_{t\to\infty} y(t) = 0$. However from (2.1) we find

$$z(t) \leqslant y(t) + p^*\,\delta\, y(\tau(t)) - h(t).$$

As $t \to \infty$ the last inequality leads to $L \leqslant 0$, which is impossible since $z(t) > 0$, $z'(t) > 0$.

Case 2.2.2. $z(t) < 0, t \geqslant t_3 \geqslant t_2$, thus $z(t) < 0$, $z'(t) > 0$, $z''(t) < 0$, $z'''(t) \geqslant 0$.

We claim that $\liminf_{t\to\infty} y(t) = 0$, otherwise $\liminf_{t\to\infty} y(t) > 0$, there exist $c > 0$ and $t_4 \geqslant t_3$ such that $y(t) \geqslant c$, $t \geqslant t_4$ from (2.2) we get

$$z'''(t) \geqslant -q^* \beta y(\sigma(t)). \tag{2.10}$$

Integrating (2.10) from t_4 to t we get

$$z''(t) - z''(t_4) \geqslant -q^*\, \beta \int_{t_4}^{t} y(\sigma(s))\, ds,$$

$$z''(t) - z''(t_4) \geqslant -q^*\, \beta\, c(t - t_4),$$

the last inequality implies that $\lim_{t\to\infty} z''(t) = \infty$, which is a contradiction. Hence $\liminf_{t\to\infty} y(t) = 0$, so by Lemma 2.1 it follows that $\lim_{t\to\infty} y(t) = 0$. □

Example 2.6. Consider the neutral differential equation

$$\left[y(t) + e^{-\frac{1}{4}} f(y(\frac{t-1}{4}))\right]''' - e^{-2} g(y(t)) = -2e^{-t} - \frac{1}{64} e^{-\frac{t}{4}}, \quad t \geqslant 1, \tag{2.11}$$

$\tau(t) = \frac{t-1}{4}$, $\sigma(t) = t$, $p(t) = e^{-\frac{1}{4}}$, $q(t) = -e^{-2}$, $F(t) = -2e^{-t} - \frac{1}{64} e^{-\frac{t}{4}}$, $f(y) = y$, $g(y(t)) = e^2 y(t)$. One can find that all conditions of Theorem 2.5 are held. To see this condition (2.6):

$$\limsup_{t\to\infty} q(t) = \lim_{t\to\infty} -e^{-2} = -0.135 < 0.$$

So, every nonoscillatory solution of (2.11) tends to zero or $|y(t)| \to \infty$ as $t \to \infty$. For instance $y(t) = e^{-t}$ is such a solution.

Theorem 2.7. *Assume that* A_1–A_5 *hold and,* $p_1 \geqslant p(t) \geqslant p_2 > \frac{1}{\delta_2}$, $q(t) \geqslant 0$, *in addition to*

$$\liminf_{t\to\infty} q(t) > 0.$$

Then every nonoscillatory solution of (1.1) *tends to zero as* $t \to \infty$.

Proof. The proof is similar to the proof of Theorem 2.3, only we used Lemma 2.2 instead of Lemma 2.1. □

Theorem 2.8. *Assume that* A_1–A_5 *hold,* $p_1 \geqslant p(t) \geqslant p_2 > \frac{1}{\delta_2}$, $q(t) \leqslant 0$ *addition to*

$$\limsup_{t\to\infty} q(t) < 0.$$

Then every nonoscillatory solution $y(t)$ *of* (1.1) *tends to zero or* $|y(t)| \to \infty$ *as* $t \to \infty$.

Proof. The proof is similar to the proof of Theorem 2.5, only we used Lemma 2.2 instead of Lemma 2.1. □

References

[1] B. Baculíková, J. Džurina, *On the asymptotic behavior of a class of third order nonlinear neutral differential equations*, Cent. Eur. J. Math., **8** (2010), 1091–1103.

[2] D. D. Baĭnov, D. P. Mishev, *Oscillation theory for neutral differential equations with delay*, Adam Hilger, Ltd., Bristol, (1991).

[3] P. Das, B. B. Mishra, C. R. Dash, *Oscillation theorems for neutral delay differential equations of odd order*, Bull. Inst. Math. Acad. Sin. (N.S.), **1** (2006), 557–568. 1

[4] J. G. Dix, *Oscillation of solutions to a neutral differential equation involving an* n*-order operator with variable coefficients and a forcing term*, Differ. Equ. Dyn. Syst., **22** (2014), 15–31.

[5] Z. Došlá, P. Liška, *Comparison theorems for third-order neutral differential equations*, Electron. J. Differential Equations, **2016** (2016), 13 pages.

[6] M. M. A. El-Sheikh, R. Sallam, N. Mohamady, *New oscillation criteria for general neutral delay third order differential equations*, Int. J. Math. Comput. Appl. Res., **3** (2013), 183–190. 1

[7] Y.-L. Fu, Y.-Z. Tian, C.-M. Jiang, T.-X. Li, *On the asymptotic properties of nonlinear third-order neutral delay differential equations with distributed deviating arguments*, J. Funct. Spaces, **2016** (2016), 5 pages.

[8] I. Győri, G. Ladas, *Oscillation theory of delay differential equations*, With applications, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, (1991). 1, 2

[9] J. Jaroš, T. Kusano, *On oscillation of linear neutral differential equations of higher order*, Hiroshima Math. J., **20** (1990), 407–419. 1

[10] C.-M. Jiang, Y. Jiang, T.-X. Li, *Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments*, Adv. Difference Equ., **2016** (2016), 14 pages.

[11] Y. Jiang, T.-X. Li, *Asymptotic behavior of a third-order nonlinear neutral delay differential equation*, J. Inequal. Appl., **2014** (2014), 7 pages. 1

[12] B. Karpuz, Ö. Öcalan, S. Öztürk, *Comparison theorems on the oscillation and asymptotic behaviour of higher-order neutral differential equations*, Glasg. Math. J., **52** (2010), 107–114. 1

[13] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, (1987).

[14] Q. Li, R. Wang, F.-W. Meng, J.-X. Han, *Asymptotic properties of solutions to third-order nonlinear neutral differential equations*, Abstr. Appl. Anal., **2014** (2014), 4 pages.

[15] H. A. Mohamad, *Oscillation of linear neutral differential equations of third order*, Iraqi J. Sci., **50** (2009), 543–548. 1

[16] H. A. Mohamad, S. N. Ketab, *Oscillation and nonoscillation properties of solutions of third order linear neutral differential equations*, Int. J. Appl. Math. Statist. Sci., **5** (2016), 31–38. 1

[17] N. Parhi, R. N. Rath, *Oscillatory behaviour of solutions of nonlinear higher order neutral differential equations*, Math. Bohem., **129** (2004), 11–27. 1

[18] R. N. Rath, L. N. Padhy, N. Misra, *Oscillation of solutions of non-linear neutral delay differential equations of higher order for* $p(t) = \pm 1$, Arch. Math. (Brno), **40** (2004), 359–366. 1