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Differential Transformation Method and Variation Iteration Method for Cauchy Reaction-diffusion Problems

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ABSTRACT

The paper reads the flow of an incompressible, unidirectional, steady third grade non-Newtonian fluid between two infinite planes. The flow is symmetric with respect to $x-$ axis with constant pressure gradient. The governing equations for the flow are second order nonlinear differential equations. Homotopy Analysis Method is applied to obtain the solution.

Keywords: Two layer flow; non-Newtonian flow; HAM solution

AMS Subject Classifications MSC: 76A05; 76M99
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1. Introduction

The simplest nonlinear flow model which describes the boundary layer flow is the Newtonian flow model described by the Navier-Stokes equations. The solution of this unsteady three dimensional equation by implementing different assumptions and boundary or/ initial conditions the number of solutions are obtained $[1-8]$. It is well known that the fluid properties are not completely described through the Navier-Stokes equations, because of the thinning and thinning effects, time dependence of the viscosities, and deformations. Therefore, non-Newtonian (differential type, rate type, integral type) flows are introduced which fulfill all these properties. Among these differential type fluids are of our interest which also exhibit the elastic effects (for small deformations), thinning-thickening effects, which are the properties of non-Newtonian fluids $[9-12]$.

A number of researchers in this direction solved the non-Newtonian fluid problems with or without boundary conditions

[13−18].

To find the solution of non-Newtonian flow problems, perturbation approach played an important role, because, with the assumption of making the non-Newtonian parameter as small the nonlinear part is perturbed and linear systems are made to solve the governing systems [19−22]. However, it was not applicable for large parameters. Liao [23−28] is the first to introduce a method which is advanced and better than the perturbation and it gives the analytic solution, namely, Homotopy Analysis Method. The beauty of the procedure is that it solves the nonlinear partial differential equations without prescribing the small or large parameter, and almost gives the exact solution.

The present paper is an attempt to study the third grade unidirectional flow between two infinite boundaries influenced by a constant pressure gradient. The fluid is flowing in x-direction and is symmetric. The flow equations arising are steady nonlinear second order differential equation. Homotopy Analysis Method is used to find the solutions of two layers separately. The graphs are also sketched to see the physical impacts of non-Newtonian parameters.

2. Problem Formulation and Governing Equations

We study the flow of two immiscible incompressible third grade fluids between two straight infinite plates under a constant pressure gradient. The flow geometry under consideration is depicted in Figure 1.

We choose the x-axis in the direction of the mid line and y-axis normal to it. The gap between these two plates is defined as $2b$ and the flow between these plates is in the x-direction. The system is symmetric with respect to the horizontal axis. The fluid is filled between two regions. In the fluid region 1, the fluid is dense, and supposed to be non-Newtonian, whereas in the fluid region 2, we assume a Newtonian fluid. The equations of flow for each fluid layer, in the absence of body forces, can be expressed as

$$div\mathbf{V}^{(k)} = 0 \quad k = 1,2 \tag{1}$$

$$\rho^{(k)} \frac{d\mathbf{V}^{(k)}}{dt} = div\mathbf{S}^{(k)} - grad(p) \tag{2}$$

Here $V^{(k)}$ is the velocity vector, $\rho^{(k)}$ is the density of each layer and the superscript k denotes the layer number. The operator d/dt denotes the material time derivative and p is the pressure. The extra stress tensor $\mathbf{S}$ for each layer is defined by

$$\mathbf{S}^{(k)} = \mu^{(k)}\mathbf{A_1}^{(k)} + \alpha_2^{(k)}\mathbf{A_1}^{(k)} + \alpha_1^{(k)}\mathbf{A_2}^{(k)} \tag{3}$$

$$+\beta_1^{(k)}\mathbf{A_3}^{(k)} + \beta_2^{(k)}\left(\mathbf{A_2}^{(k)}\mathbf{A_1}^{(k)} + \mathbf{A_1}^{(k)}\mathbf{A_2}^{(k)}\right) + \beta_3^{(k)}(tr\mathbf{A_2}^{(k)})\mathbf{A_1}^{(k)}, \;\; k = 1,2.$$

In equation (3), $\mu^{(k)}$ is the coefficient of viscosities and $\alpha_1^{(k)}$, $\alpha_1^{(k)}$, $\beta_1^{(k)}$, $\beta_2^{(k)}$ and $\beta_3^{(k)}$ are material constants for fluids 1 and 2. The Rivlin-Ericksen tensors $\mathbf{A_n}^{(k)}$ are defined by the relation

$$\mathbf{A_n}^{(k)} = \frac{d}{dt}\mathbf{A_{n-1}}^{(k)} + \mathbf{A_{n-1}}^{(k)}(grad\mathbf{V}^{(k)}) + (grad\mathbf{V}^{(k)})\mathbf{A_{n-1}}^{(k)}, \quad n > 1 \tag{4}$$

$$\mathbf{A_1}^{(k)} = (grad\mathbf{V}^{(k)}) + (grad\mathbf{V}^{(k)})^T \tag{5}$$

The flow is steady and fully developed, and the velocity fields and the extra stress are assumed to be the following form:

$$\mathbf{V}^{(k)} = [u^{(k)}(y), 0, 0] \tag{6}$$

$$\mathbf{S}^{(k)} = \mathbf{S}^{(k)}(y) \tag{7}$$

By substitution these relations into the field equations, we find that equations (1) is satisfied identically and equations (2) takes the following form:

$$\rho^{(k)}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = grad\begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix} + div\begin{pmatrix} S_{xx}^{(k)}(y) & S_{xy}^{(k)}(y) \\ S_{yx}^{(k)}(y) & S_{yy}^{(k)}(y) \end{pmatrix} \tag{8}$$

where the components of extra stress $\mathbf{S}^{(k)}$ are calculated separately by making use of equations (5) in (4) and (3). Equation (6) in component form can be written as

$$0 = -\frac{\partial p}{\partial x} + \frac{d}{dy} S_{xy}^{(k)} \tag{9}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{d}{dy} S_{yy}^{(k)} \tag{10}$$

If we introduce generalized pressure $\widehat{p}$ as

$$\widehat{p} = p - S_{yy}^{(k)} \tag{11}$$

we may rewrite the above system as

$$0 = -\frac{\partial p}{\partial x} + \frac{d}{dy} S_{xy}^{(k)} \tag{12}$$

$$0 = -\frac{\partial p}{\partial y} \tag{13}$$

Equation (11) implies that $\widehat{p}$ is independent of y and accordingly the equation (10) reduces to

$$\frac{d}{dy} S_{xy}^{(k)} = \frac{\partial p}{\partial x} = -G \tag{14}$$

The expression for $S_{xy}^{(k)}$ in combination with (12) yields the following differential equation:

$$\mu^{(k)} \frac{d^2 u^{(k)}}{dy^2} + 6\left(\beta_2^{(k)} + \beta_3^{(k)}\right)\left(\frac{du^{(k)}}{dy}\right)^2 \frac{d^2 u^{(k)}}{dy^2} = -G, \quad k = 1, 2 \tag{15}$$

On letting

$$\beta_2^{(k)} + \beta_3^{(k)} = \beta^{(k)}, \quad k = 1, 2 \tag{16}$$

we rewrite (15) as

$$\mu^{(k)} \frac{d^2 u^{(k)}}{dy^2} + 6\beta^{(k)} \left(\frac{du^{(k)}}{dy} \right)^2 \frac{d^2 u^{(k)}}{dy^2} = -G, \quad k = 1, 2 \tag{17}$$

Thus the resulting differential equations for each layer are

$$\mu^{(1)} \frac{d^2 u^{(1)}}{dy^2} + 6\beta^{(1)} \left(\frac{du^{(1)}}{dy} \right)^2 \frac{d^2 u^{(1)}}{dy^2} = -G \tag{18}$$

$$\mu^{(2)} \frac{d^2 u^{(2)}}{dy^2} + 6\beta^{(2)} \left(\frac{du^{(2)}}{dy} \right)^2 \frac{d^2 u^{(2)}}{dy^2} = -G \tag{19}$$

2.1 The Boundary Conditions

The boundary conditions associated with the problem under consideration are:

(a) Shear stress at the interface is the same for both fluids. That is

$$at \quad y = 0 \quad S_{xy}^{(1)} = S_{xy}^{(2)} \tag{20}$$

(b) Velocity is continuous at the interface of both fluids. That is

$$at \quad y = 0 \quad u^{(1)} = u^{(2)} \tag{21}$$

(c) No slip at the channel walls. That is

$$\begin{aligned} &at \quad y = -b \quad u^{(1)} = 0 \\ &at \quad y = b \quad u^{(2)} = 0 \end{aligned} \tag{22}$$

We point out that equations (18) and (19) are two non-linear second order differential equations and the exact solution subject to boundary conditions $(20)-(22)$ seems to be impossible. To solve this we apply the Homotopy Analysis Method (HAM) as introduced by Liao $[23-28]$.

3. HAM Solution

The base function for equations (18) and (19) is

$$\{y^m, m \geq 1\} \tag{23}$$

The solution for the equations can be written as

$$u^1 = \sum_{m=1}^{+\infty} a_m y^m, \tag{24}$$

$$u^2 = \sum_{m=1}^{+\infty} b_m y^m. \tag{25}$$

The initial solutions for the equation (18) and (19) are

$$u_0^1 = \frac{b^2 G}{2\mu^{(1)}}\left[m_1 + n\left(\frac{y}{b}\right) - \left(\frac{y}{b}\right)^2 \right], \tag{26}$$

$$u_0^2 = \frac{b^2 G}{2\mu^{(2)}}\left[m_2 + n\left(\frac{y}{b}\right) - \left(\frac{y}{b}\right)^2 \right], \tag{27}$$

where

$$m_1 = \frac{2\mu^{(1)}}{\mu^{(1)} + \mu^{(2)}}, \qquad m_2 = \frac{2\mu^{(2)}}{\mu^{(1)} + \mu^{(2)}}, \qquad n = \frac{\mu^{(1)} - \mu^{(2)}}{\mu^{(1)} + \mu^{(2)}}.$$

The linear operators chosen are

$$\mathbf{L}_1[\phi_1(y;q)] = \frac{\partial^2 \phi_1}{\partial y^2}, \tag{28}$$

$$\mathbf{L}_2[\phi_2(y;q)] = \frac{\partial^2 \phi_2}{\partial y^2}, \tag{29}$$

$$\mathbf{L}_1[C_1 y + C_2] = 0, \tag{30}$$

$$\mathbf{L}_2[C_3 y + C_4] = 0, \tag{31}$$

where C_1, C_2, C_3 and C_4 are integral constants. The non-linear operators are

$$\mathbf{N}_1[\phi_1(y;q)] = \mu^{(1)} \frac{d^2\phi_1}{dy^2} + 6\beta^{(1)}\left(\frac{d\phi_1}{dy}\right)^2\left(\frac{d^2\phi_1}{dy^2}\right) + G, \tag{32}$$

$$\mathbf{N}_2[\phi_2(y;q)] = \mu^{(2)} \frac{d^2\phi_2}{dy^2} + 6\beta^{(2)}\left(\frac{d\phi_2}{dy}\right)^2\left(\frac{d^2\phi_2}{dy^2}\right) + G, \tag{33}$$

The HAM deformation equations are

$$\mathbf{L}_1[\phi_1(y,q) - u_0^1] = qh\mathbf{N}_1[\phi_1(y;q)], \tag{34}$$

$$\mathbf{L}_2[\phi_2(y,q) - u_0^2] = qh\mathbf{N}_2[\phi_2(y;q)], \tag{35}$$

and the corresponding boundary conditions are

$$\phi_1(0,q)=\phi_2(0,q), \tag{36}$$

$$\phi_1(-b,q)=0, \tag{37}$$

$$\mu^{(1)}\frac{d\phi_1}{dy}\bigg|_{y=0}=\mu^{(2)}\frac{d\phi_2}{dy}\bigg|_{y=0}, \tag{38}$$

$$\phi_2(b,q)=0, \tag{39}$$

where $q\varepsilon[0,1]$ is an embedding parameter, when $q=0$ then from (34) to (37) we get

$$\phi_1(y,0)=u_0^1, \tag{40}$$

$$\phi_2(y,0)=u_0^2, \tag{41}$$

when $q=1$

$$\phi_1(y,1)=u^1, \tag{42}$$

$$\phi_2(y,1)=u^2, \tag{43}$$

using (40,41), we expand $\phi_1(y,q)$ and $\phi_2(y,q)$ in the Taylor Serious with respect to q, i.e

$$\phi_1(y,q)=u_0^1+\sum_{m=1}^{\infty}u_m^1q^m, \tag{44}$$

$$\phi_2(y,q)=u_0^2+\sum_{m=1}^{\infty}u_m^2q^m, \tag{45}$$

where

$$u_m^1=\frac{1}{m!}\frac{\partial^m\phi_1(y;q)}{\partial q^m}\bigg|_{q=0}, \tag{46}$$

$$u_m^2=\frac{1}{m!}\frac{\partial^m\phi_2(y;q)}{\partial q^m}\bigg|_{q=0}. \tag{47}$$

The above series convergence at q 🖬 1

$$u^1=u_0^1+\sum_{m=1}^{\infty}u_m^1, \tag{48}$$

$$u^2=u_0^2+\sum_{m=1}^{\infty}u_m^2. \tag{49}$$

4. Higher-Order Deformation Equations

$$\overleftrightarrow{u^1}_m = \{u_0^1, u_1^1, u_2^1, \ldots, u_n^1\}, \tag{50}$$

$$\overleftrightarrow{u^2}_m = \{ u_0^2, u_1^2, u_2^2, \ldots, u_m^2 \}. \tag{51}$$

Differentiating the HAM deformation equations (34) and (35) m and n times with respect to q, then setting $q=0$ and dividing by $m!$, we have the m th-order deformation equations

$$\mathbf{L}_1[u_m^1 - \chi_m u_{m-1}^1] = hR_m(\overleftrightarrow{u^1}_{m-1}), \tag{52}$$

$$\mathbf{L}_2[u_m^2 - \chi_m u_{n-1}^2] = hR_m(\overleftrightarrow{u^2}_{m-1}), \tag{53}$$

where

$$R_m^{u^1} = \mu^{(1)} u''^1_{m-1} + 6\beta^{(1)} \sum_{j=0}^{m-1} u''^1_j \sum_{i=0}^{m-1-j} u'^1_i u'^1_{m-1-j-i} + G(1-\chi_m), \tag{54}$$

$$R_m^{u^2} = \mu^{(2)} u^{2''}_{m-1} + 6\beta^{(2)} \sum_{j=0}^{m-1} u^{2''}_j \sum_{i=0}^{m-1-j} u^{2'}_i u^{2'}_{m-1-j-i} + G(1-\chi_m), \tag{55}$$

subject to the boundary conditions

$$\mu^{(1)} \frac{\partial^m u'^1}{\partial q^m}\bigg|_{q=0,\, y=0} = \mu^{(2)} \frac{\partial^m u'^2}{\partial q^m}\bigg|_{q=0,\, y=0}, \tag{56}$$

$$u_m^1(0) = u_m^2(0), \tag{57}$$

$$u_m^1(-b) = 0, \quad u_m^2(b) = 0, \tag{58}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{59}$$

5. Result and Discussion

The figures ($2-5$) are plotted for different values of β^1 and β^2 $(0.05-0.1)$; $\mu^1(1/2)$, $\mu^2(1/5)$; $G(1/20)$ and $\hbar(-3/2)$. The figue 2 is sketched for the $\hbar$ curve that provides us the convergence region i.e. $0 < \hbar < -4$. The figure 3 is sketched for $\beta^1 = 1/5$, $\beta^2 = 1/10$; $\mu^1 = 1/2)$, $\mu^2 = 1/5$; $G(1/20)$; $\hbar(-3/2)$ and it shows the convergence for u^2 with $20th$, $30th$ and $40th$ order approximations. In figures 4 and 5, we assume different values of $\beta^1\,(1/5;1/10;1/100)$ and

$\beta^2 (1/10; 1/100; 1/1000)$ and provide the $20th$ order result for u^1 and u^2. It is observed that with an increase in the third grade parameters the velocity boundary layer thickness decreases. It is worth mentioning here that the solutions for both the cases exist and converge $[22-28]$, and also these satisfy all the previous findings (regarding the third grade fluids) $[9-21]$.

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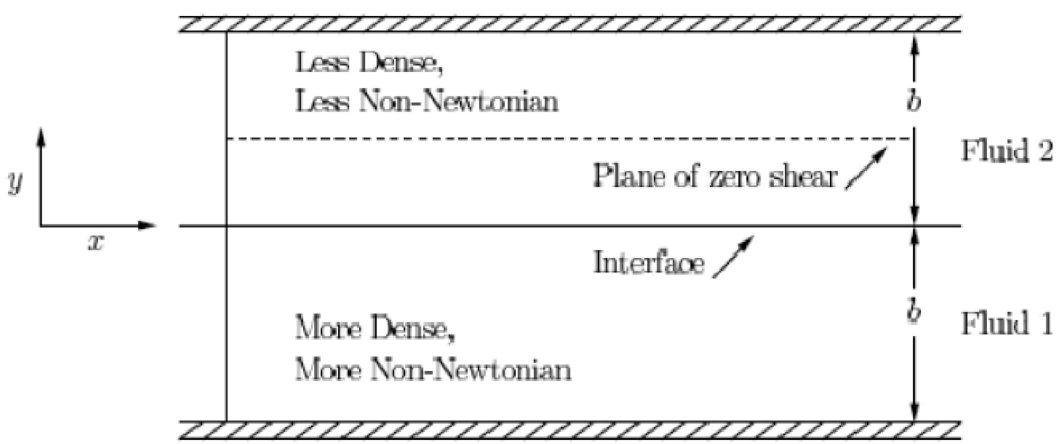

Figure 1. Geometry of the problem

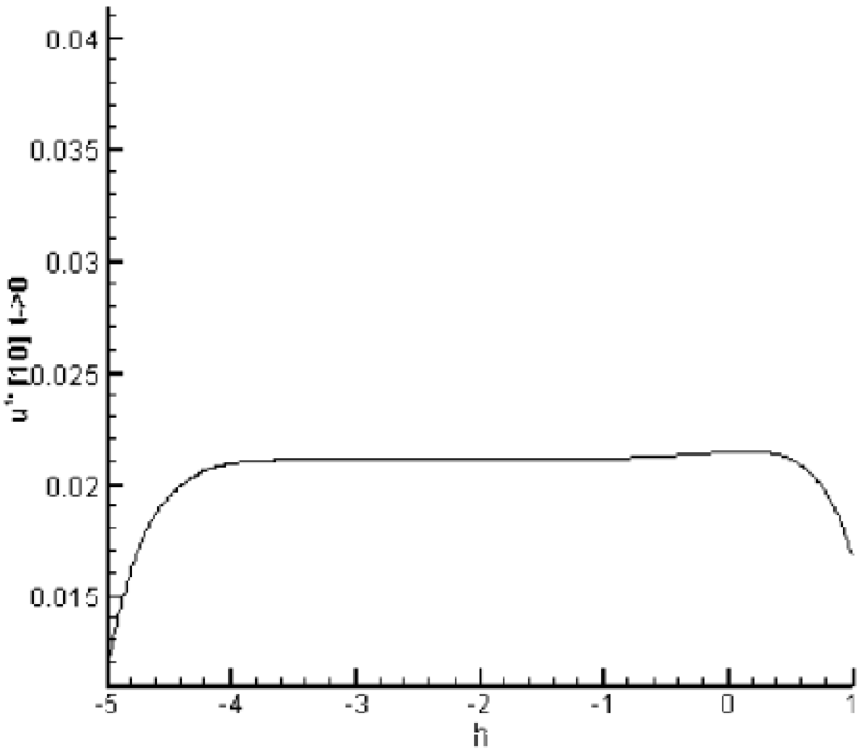

Figure 2. $\hbar$ curve

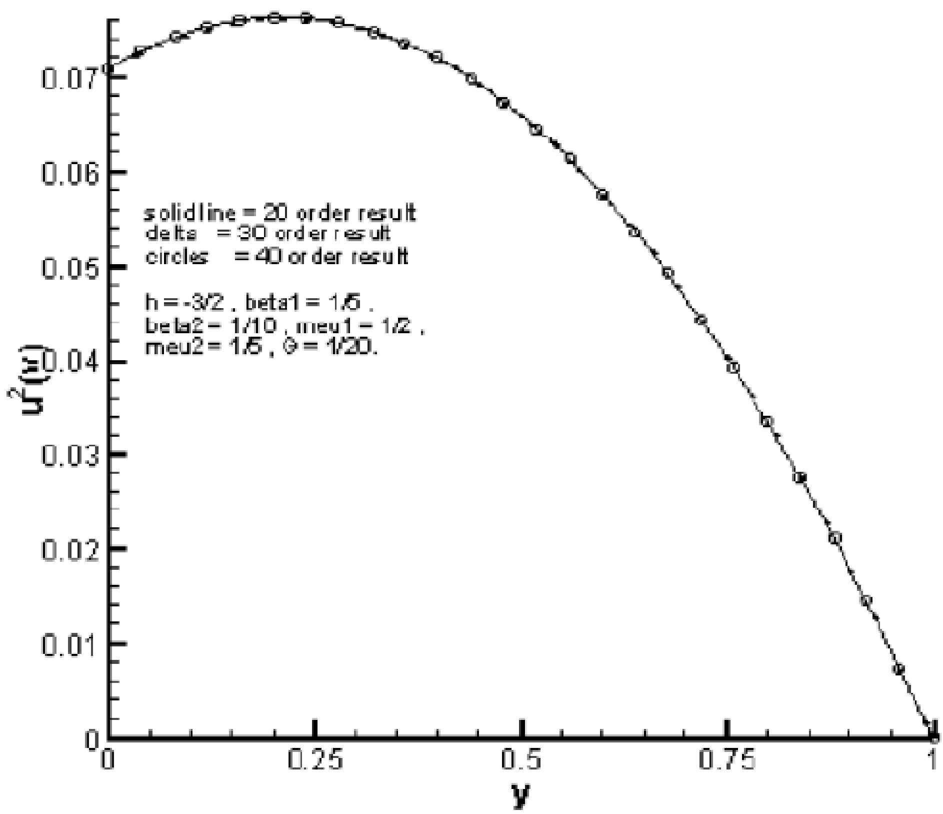

Figure 3. The convergence of $u^{(2)}$ with $\hbar = -3/2$

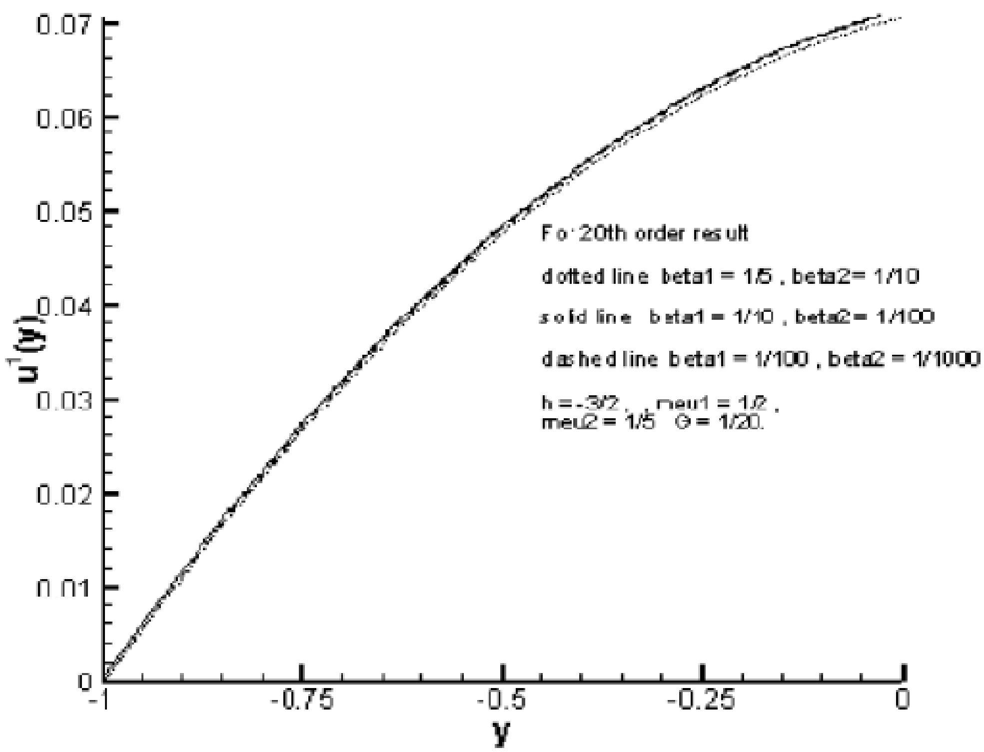

Figure 4. The sketch of $u^{(1)}$ for different values of $\beta^{(1)}$ and $\beta^{(2)}$

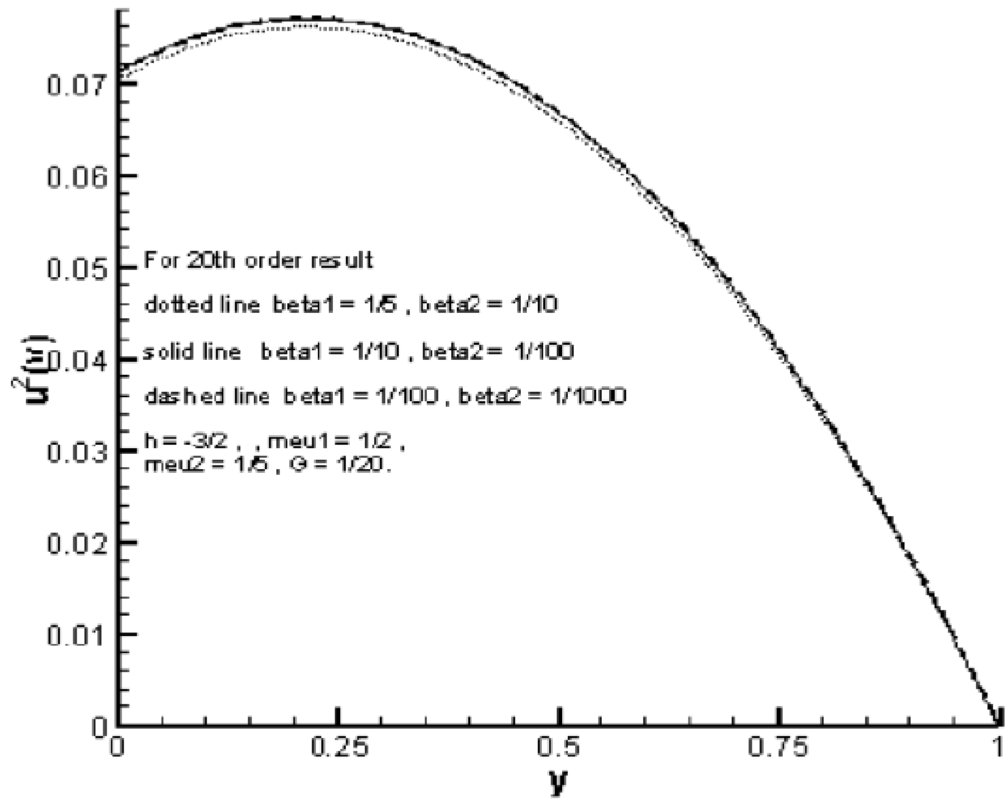

Figure 5. The sketch of $u^{(2)}$ for different values of $\beta^{(1)}$ and $\beta^{(2)}$

Figure Captions

Figure 1. Geometry of the problem

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