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Differential Transformation Method and Variation Iteration Method for Cauchy Reaction-diffusion Problems

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Abstract

In this chapter, we will compare the differential transform method (DTM) and variational iteration method (VIM) for solving the one-dimensional, time dependent reaction-diffusion equations. Different cases of the equation are discussed and analytical solution in series form can be derived. The results obtained by the proposed method (DTM) are compared with the results obtained by (VIM). Some examples are presented to show the ability of the methods for such problems.

Keywords: Differential transformation, Variation iteration method, Cauchy reaction- diffusion problems, Taylor's series expansion**.**

1. Introduction

This paper outlines a reliable comparison between two powerful methods that were recently developed. The first is the differential transformation method (DTM) which was first proposed by Zhou (1986), [47], who solved linear and non-linear initial value problems in electric Circuit analysis, and was used heavily in the literature successfully applied to eigenvalue problems [7,17], linear and non-linear higher-order boundary value problems [13], one-dimensional planar Bratu problem [12], higher-order initial value problems [9, 34], systems of ordinary and partial differential equations [5, 11], high index differential-algebraic equations [6, 39], integro-differential equations [3], and non-linear oscillators. The second is variational iteration method (VIM) which was first proposed by He [22, 23]. He [25, 32] has succeeded in applying (VIM) to autonomous ordinary differential equations, non-linear systems of partial differential equations [26], and construction of solitary solutions. He has also succeeded in applying compaction-like solutions to partial differential equations [31]. Other scientists who applied those two methods to other fields are listed in [1, 2, 15, 24, 29, 41-43, 46].

In this section, we consider the one-dimensional Cauchy reaction-diffusion problems

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + p(x,\ t)u(x,t), \quad (x,t) \in \Omega \subset R^2. \tag{1}$$

where u is the concentration, p is the reaction parameter and $D > 0$ is the diffusion coefficients, are subject to the initial and boundary conditions

$$u(x,0) = g(x), \quad x \in R. \tag{2}$$

$$u(0,t) = f_0(t), \quad \frac{\partial u}{\partial x}(0,t) = f_1(t). \tag{3}$$

The problem given by Eqs. (1) and (2) is called the characteristic Cauchy problem in domain $\Omega = R \times R_+$, whilst the problem given by Eqs. (1) and (3) is called the non-characteristic Cauchy problem in the domain $\Omega = R \times R_+$. The solution of these problems is attempted by using the differential transformation method (DTM) and variation iteration method (VIM), This will be discussed later.

We will now describe the model of the problems that will be used for our analysis experiment.

Problem 1: Consider the Eq. (1) when $p(x,t) = \text{const},\ t = -1.$ In this case, we put the problem in the form:

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) - u(x,t), \quad (x,t) \in \Omega \subset R^2. \tag{4}$$

with the initial and boundary conditions

$$u(x,0) = e^x + x, \quad x \in R. \tag{5}$$

$$u(0,t) = 1 \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = e^{-t} - 1, \quad t \in R. \tag{6}$$

Problem 2: Consider the Eq. (1) when $p(x,t) = p(t)$ only where $p(t) = 2t.$ In this case, we have the problem in the form:

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + 2\,t u(x,t), \quad (x,t) \in \Omega \subset R^2. \tag{7}$$

with the initial and boundary conditions

$$u(x,0) = e^x, \quad x \in R. \tag{8}$$

$$u(0,t) = e^{(t+t^2)} \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = e^{(t+t^2)}, \quad t \in R. \tag{9}$$

Problem 3: Consider the Eq. (1) when $p(x,t) = p(x)$ only where $p(x) = -1 - 4x^2.$ In this case, we have the problem in the form:

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + (-1 - 4x^2)u(x,t), \quad (x,t) \in \Omega \subset R^2. \tag{10}$$

with the initial and boundary conditions

$$u(x,0) = e^{x^2}, \quad x \in R. \tag{11}$$

$$u(0,t) = e^{t} \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad t \in R. \tag{12}$$

Problem 4: Consider the Eq. (1) when $p(x,t) = p(x,t)$ where $p(x,t) = 2t - 2 - 4x^2$. In this case, we have the problem in the form:

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + (2t - 2 - 4x^2)u(x,t), \quad (x,t) \in \Omega \subset R^2. \tag{13}$$

with the initial and boundary conditions

$$u(x,0) = e^{x^2}, \quad x \in R. \tag{14}$$

$$u(0,t) = e^{t^2} \quad \text{and} \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad t \in R. \tag{15}$$

Our work, in this paper, relies mainly on two of the most recently methods, the (DTM) and (VIM). The two methods, which accurately compute the solutions in a series form or in an exact form, are of great interest to applied sciences. The effectiveness and the usefulness of both methods are demonstrated by finding exact solutions to the above problems 1-4, that will be investigated. However, each method has its own characteristics and significance that will be examined.

2. Basic of Differential Transformation Method

In what follows we will highlight briefly the main points of the methods. The details can be found in [4, 8, 10, 12, 16, 18, 19, 20, 35, 37].

(i) One-dimensional differential transformation:

The basic definitions of the differential transformation are introduced as follows:

Definition 1: An k^{th}-order differential transformation (DT) of a function $y(x) = f(x)$ is defined when a point $x = x_0$ as:

$$Y(x) = \frac{1}{k!}[\frac{d^k}{dx^k} y(x)]_{x=0}. \tag{16}$$

where k belongs to the set of non-negative integers, denoted as the K-domain.

Definition 2: The function $y(x)$ may be expressed in terms of the differential transformation (DT), $Y(x)$ as:

$$y(x) = \sum_{k=0}^{\infty} (x - x_0)^k Y(k). \tag{17}$$

Upon combining (16) and (17), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} \{\frac{d^k}{dx^k} y(x)\}|_{x=0}. \tag{18}$$

which is actually the Taylor's series for $y(x)$ when $x = x_0$.

(ii) Two-dimensional differential transformation

Similarly, we consider a function of two variables $w(x,y)$ analytic in the domain K and let $(x,y)=(x_0,y_0)$ in this domain. The function $w(x,y)$ is then represented by one power series whose center at located is (x_0,y_0). The differential transformation of function $w(x,y)$ is the form

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y^h}w(x,y)\right]_{(x_0,y_0)}. \qquad (19)$$

where $w(x,y)$ is the original function and $W(k,h)$ is the transformed function.

The differential inverse transform of $W(k,h)$ is defined as:

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(y-y_0)^h. \qquad (20)$$

From Eqs. (19) and (20) we can conclude:

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!}\frac{1}{h!}\{\frac{\partial^{k+h}}{\partial x^k \partial y^h}w(x,y)\}\Big|_{(x_0,y_0)}(x-x_0)^k(y-y_0)^h. \qquad (21)$$

When we apply Eqs. (20) at $(x_0,y_0)\equiv(0,0)$, then (20) can be written by a finite series as:

$$w(x,y)=\sum_{k=0}^{M}\sum_{h=0}^{N}W(k,h)x^k y^h. \qquad (22)$$

The application of the differential transformation method (DTM) will be discussed, for solving problems 1-4. According to the (DTM) and the operations mathematics of the method, we consider Eq. (1) after taking the differential transformation of both sides in the following form:

$$U(k,h+1)=\frac{1}{(h+1)}\left[D(k+1)(k+2)U(k+2,h)+\sum_{r=0}^{k}\sum_{s=0}^{k}P(r,h-s)U(k-r,s)\right]. \qquad (23)$$

By taking the differential transform to the initial (boundary) conditions (2) and (3) respectively, we get

$$U(k,0)=G(k), \qquad (24)$$

$$U(0,h)=F_0(h), \qquad (25)$$

$$U(1,h)=F_1(h), \qquad (26)$$

By using (24)-(26) into (23), we can calculate $U(k,h)$. When we substitute all $U(k,h)$ into (22) as $M\to\infty$ and $N\to\infty$, the solution $U(x,t)$ will be consequently readily obtained.

3. Basic Ideas of He's Variational Iteration Method

In this section, the variational iteration method will be applied for solving problems 1-4. According to the variational iteration method, where details can be found in [14, 15, 21, 25-28, 31, 33, 36, 44, 45], we consider the correction functional for Eq. (1) in the form:

$$U_{n+1}(x,t)=u(x,t)+\int_0^t\lambda(s)[\frac{\partial u_n}{\partial s}(x,s)-D\frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,s)-P(x,s)u_n(x,s)]ds. \qquad (27)$$

where λ is the general Lagrange multiplier [33], u_0 is an initial approximation which must be chosen suitably and $\tilde{u}_n$ is the restricted variation, i.e. $\delta \tilde{u}_n = 0$ as in [22-25]. To find the optimal value of λ we have

$$\delta U_{n+1}(x,t) = \delta u(x,t) + \delta\int_0^t \lambda(s)[\frac{\partial u_n}{\partial s}(x,s) - D\frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,s) - P(x,s)u_n(x,s)]ds, \tag{28}$$

or

$$\delta U_{n+1}(x,t) = \delta u(x,t) + \delta\int_0^t \lambda(s)[\frac{\partial u_n}{\partial s}(x,s) - P(x,s)u_n(x,s)]ds, \tag{29}$$

Therefore we have

$$\delta U_{n+1}(x,t) = \delta u(x,t)(1+\lambda) - \int_0^t \delta u_n(x,s)[\frac{\partial \lambda}{\partial x}(x,s) + P(x,s)u_n(x,s)]ds = 0, \tag{30}$$

which yields

$$\frac{\partial \lambda}{\partial x}(s) + P(x,s) = 0, \tag{31}$$

$$1 + \lambda(s) = 0\big|_{s=t}, \tag{32}$$

Thus we have

$$\lambda(s) = \int p(x,s)\,ds - \int p(x,t)\,dt - 1. \tag{33}$$

and we obtain the following iteration formula:

$$U_{n+1}(x,t) = u_n(x,t) + \int_0^t [\int p(x,s)ds - \int p(x,t)dt - 1][\frac{\partial u_n}{\partial s}(x,s) - D\frac{\partial^2 u_n}{\partial x^2}(x,s) - P(x,s)u_n(x,s)]ds. \tag{34}$$

and for sufficiently large values of n we can consider u_n as an approximation of the exact solution.

4. Test problems

To give a clear overview of the analysis introduced above, four illustrative examples have been selected to demonstrate the efficiency of the method.

Example 1: In this example we solve problem 1, when $D = 1,\ p(x,t) = -1.$

(i) By using Differential Transformation Method **(DTM)**,

When taking the differential transformation of (4), we can obtain:

$$U(k,h+1) = \frac{1}{(h+1)}\left[(k+1)(k+2)U(k+2,h) - U(k,h)\right]. \tag{35}$$

From the initial condition (5), we can write

$$\sum_{r=0}^{\infty} U(k,0)x^r = (1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots) + \delta(k-1)\delta(h), \tag{36}$$

which gives

$$U(k,0)=\begin{cases}1, & \text{if } k=0\\ 0, & \text{if } k=1\\ \dfrac{(-1)^k}{k!}, & \text{if } k=2,3,4,\cdots\end{cases}. \qquad (37)$$

From the boundary condition (6), we have

$$U(0,h)=\delta(k)\delta(h),$$

$$U(0,h)=\begin{cases}1, & \text{if } k=h=0\\ 0, & \text{if otherwise.}\end{cases} \qquad (38)$$

and

$$\sum_{r=0}^{\infty}U(1,h)t^r=(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\cdots)-\delta(k)\delta(h), \qquad (39)$$

which gives

$$U(k,0)=\begin{cases}0, & \text{if } k=0,h=0\\ -1, & \text{if } k=1,h=1\\ \dfrac{(-1)^h}{h!}, & \text{if } k=2,3,4,\cdots\end{cases}. \qquad (40)$$

For each k,h substituting Eqs. (37), (38) and (40) into Eq. (35) and by recursive method, all other of $U(k,h)$ are equal to zero. when we substitute all values $U(k,h)$ into (22) as $M\to\infty$ and $N\to\infty$, we obtain series for $u(x,t)$. Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)\,x^k t^h,$$

$$u(x,t)=(1-xt+\frac{1}{2}xt^2-\frac{1}{6}xt^3+\frac{1}{24}xt^4-\frac{1}{120}xt^5+\cdots$$
$$+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots),$$

$$u(x,t)=x\,(1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\cdots)$$
$$+(1-x-\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots),$$

$$u(x,t)=x\,e^{-t}+e^{-x}. \qquad (41)$$

(ii) By using Variational Iteration Method **(VIM)**,

Using the variational iteration method for solving problem 1, if $D=1$, the Lagrange's multiplier $\lambda(s)$ is given by

$$\lambda(s)=t-s-1. \qquad (42)$$

Substituting the value $\lambda(s)$ of (42) into the functional (34) gives the iteration formula:

$$U_{n+1}(x,t) = u_n(x,t) + \int_0^t (t-s-1)[\frac{\partial u_n}{\partial s}(x,s) - \frac{\partial^2 u_n}{\partial x^2}(x,s) + u_n(x,s)]ds. \qquad (43)$$

We can select $u_0(x,t) = \exp(-x) + x$ by using the given initial value. Accordingly, we obtain the following successive approximations:

$$u_0(x,t) = e^{-x} + x,$$

$$u_1(x,t) = e^{-x} + x(1 - t + \frac{1}{2}t^2),$$

$$u_3(x,t) = e^{-x} + x(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4),$$

$$u_4(x,t) = e^{-x} + x(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 - \frac{1}{5!}t^5 + \frac{1}{6!}t^6),$$

$$\vdots$$

Recall that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

Consequently, the exact solution is the form:

$$u(x,t) = e^{-x} + x\,e^{-t}. \qquad (44)$$

From Eqs. (42) and (44), the approximate solution of the given problem 1 by using differential transformation method is the same results as that obtained by the variational iteration method and by the Adomian decomposition method [38] respectively and it clearly appears that the approximate solution remains closed form to exact solution.

Example 2: In this example we solve problem 2, when $D = 1$ and $p(x,t) = 2t$.

(i) By using Differential Transformation Method **(DTM)**,

Taking the differential transformation of (7), we have

$$U(k,h+1) = \frac{1}{(h+1)}\left[(k+1)(k+2)U(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,h-s-1)U(k-r,s)\right]. \qquad (45)$$

From the initial condition (8), we can write

$$U(k,0) = \begin{cases} 1, & \text{if } k = m \text{ and } h = n \\ 0, & \text{if } k = 2,3,4,\cdots \text{ and } h = 0. \end{cases} \qquad (46)$$

From the boundary condition (9), consequently we can calculate the following values in Table 1.

Table 1.

$$U(0,h) = U(1,h), \quad h = 0,1,2,\cdots$$

$U(0,0)=U(1,0)=1$	$U(0,0)=U(1,1)=1$	$U(0,2)=U(1,2)=\frac{3}{2}$
$U(0,3)=U(1,3)=\frac{7}{6}$	$U(0,4)=U(1,4)=\frac{25}{24}$	$U(0,5)=U(1,5)=\frac{27}{40}$
$U(0,6)=U(1,6)=\frac{331}{720}$	$U(0,7)=U(1,7)=\frac{1304}{5040}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$

Using the values in Table 1 into (45), and by recursive method, we have

$$U(k,1)=\frac{1}{k!},\quad k=2,3,4,\cdots \tag{47}$$

and

$$U(2,2)=\frac{3}{4},\ \ U(3,2)=\frac{1}{4},\ \ U(4,2)=\frac{1}{16},\ \ U(5,2)=\frac{1}{180},\ \ U(6,2)=\frac{1}{480},\cdots,$$
$$U(2,3)=\frac{7}{12},\ \ U(3,3)=\frac{7}{36},\ \ U(4,3)=\frac{7}{144},\cdots,\ \ U(2,4)=\frac{25}{48},\ \ U(3,4)=\frac{55}{144},\cdots,$$
$$U(2,5)=\frac{27}{80},U(3,5)=\frac{9}{80},U(4,5)=\frac{9}{32}\cdots, \tag{48}$$

and so on, when we substitute all values $U(k,h)$ into (22) as $M\to\infty$ and $N\to\infty$, we obtain series for $u(x,t)$. Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)\,x^k t^h=e^{(x+t+t^2)}. \tag{49}$$

(ii) By using Variational Iteration Method **(VIM)**,

Using the variational iteration method for solving problem 2, when $D=1$, $p=2t$. The Lagrange's multiplier $\lambda(s)$ is given by

$$\lambda(s)=s^2-t^2-1. \tag{50}$$

Substituting the value $\lambda(s)$ of (50) into the functional (34) gives the iteration formula:

$$U_{n+1}(x,t)=u_n(x,t)+\int_0^t(s^2-t^2-1)[\frac{\partial u_n}{\partial s}(x,s)-\frac{\partial^2 u_n}{\partial x^2}(x,s)+u_n(x,s)]ds. \tag{51}$$

We can select $u_0(x,t)=\exp(x)$ by using the given initial value. Accordingly, we obtain the following successive approximations:

$$u_0(x,t)=e^x,$$
$$u_1(x,t)=e^x(1+t+t^2+\frac{2}{3}t^3+\frac{1}{4}t^4),$$
$$u_2(x,t)=e^x(1+t+\frac{3}{2}t^2+t^3+\frac{11}{12}t^4+\frac{1}{2}t^5+\frac{2}{9}t^6+\frac{11}{105}t^7+\frac{1}{24}t^8),$$

$$u_3(x,t) = e^x(1 + t + \frac{3}{2}t^2 + \frac{7}{6}t^3 + t^4 + \frac{13}{20}t^5 + \frac{5}{12}t^6 + \frac{25}{126}t^7 + \frac{139}{1260}t^8 + \frac{197}{4536}t^9 + \frac{173}{12600}t^{10} + \frac{211}{41580}t^{11} + \frac{1}{720}t^{12}),$$

$$\vdots$$

Recall that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

Consequently, the exact solution is the form:

$$u(x,t) = e^x \times e^{t+t^2}. \qquad (52)$$

From Eqs. (50) and (52), the approximate solution of the given problem 2 by using differential transformation method gives the same results as that obtained by the variational iteration method and by the Adomian decomposition method [38] respectively and it clearly appears that the approximate solution remains closed form to exact solution.

Example 3: In this example we solve problem 3, when $D = 1$ and $p(x,t) = -1 - 4x^2$.

(i) By using Differential Transformation Method **(DTM)**,

Taking the differential transformation of (10), we have

$$U(k,h+1) = \frac{1}{(h+1)}\begin{bmatrix}(k+1)(k+2)U(k+2,h) - U(k,h) \\ -\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-2,h-s)U(k-r,s)\end{bmatrix}. \qquad (53)$$

From the initial condition (11), we can write

$$U(k,0) = \begin{cases} 0, & \text{if } k = 1,3,5\cdots \\ \dfrac{1}{(\frac{k}{2})!}, & \text{if } k = 0,2,4,6\cdots \end{cases} \quad \text{and } h = 0. \qquad (54)$$

From the boundary condition (12), consequently we can write

$$U(0,h) = \frac{1}{h!}, \quad \text{forall} \quad h \geq 0. \qquad (55)$$

$$U(1,h) = 0, \quad \text{forall} \quad h \geq 0. \qquad (56)$$

Using (54)-(56) into (53) by recursive method, we have the following values in Table 2 with $U(k,h) = 0 \quad \text{for} \quad k = 3,5,7,\cdots$ and $h \geq 1$.

Table 2

$U(2,1)=1$	$U(4,1)=\frac{1}{2}$	$U(6,1)=\frac{1}{6}$	$U(8,1)=\frac{1}{24}$
$U(2,2)=\frac{1}{2}$	$U(4,2)=\frac{1}{4}$	$U(6,2)=\frac{1}{12}$	$U(8,2)=\frac{1}{48}$
$U(2,3)=\frac{1}{6}$	$U(4,3)=\frac{1}{12}$	$U(6,3)=\frac{1}{36}$	$U(8,3)=\frac{1}{144}$
$U(2,4)=\frac{1}{24}$	$U(4,4)=\frac{1}{48}$	$U(6,4)=\frac{1}{144}$	$U(8,4)=\frac{1}{576}$
$U(2,5)=\frac{1}{120}$	$U(4,5)=\frac{1}{240}$	$U(6,5)=\frac{1}{720}$	$U(8,5)=\frac{1}{2880}$
$U(2,6)=\frac{1}{720}$	$U(4,6)=\frac{1}{1440}$	$U(6,6)=\frac{1}{4320}$	$U(8,6)=\frac{1}{17280}$
$U(2,7)=\frac{1}{5040}$	$U(4,7)=\frac{1}{10080}$	$U(6,7)=\frac{1}{30240}$	$U(8,7)=\frac{1}{120960}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

when we substitute all values $U(k,h)$ into (22) as $M\to\infty$ and $N\to\infty$, we obtain series for $u(x,t)$. Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h=e^{(x^2+t)}. \tag{57}$$

(ii) By using Variational Iteration Method **(VIM)**,

Using the variational iteration method for solving problem 3, when $D=1$, $p(x,t)=(-1-4x^2)$, The Lagrange's multiplier $\lambda(s)$ is given by

$$\lambda(s)=(1+4x^2)\,t-(1+4x^2)\,s-1. \tag{58}$$

When we substitute the value $\lambda(s)$ of (58) into the functional (34), we get the iteration formula:

$$U_{n+1}(x,t)=u_n(x,t)+\int_0^t((1+4x^2)t-(1+4x^2)s-1)[\frac{\partial u_n}{\partial s}(x,s)-\frac{\partial^2 u_n}{\partial x^2}(x,s)+u_n(x,s)]ds. \tag{59}$$

We can select $u_0(x,t)=\exp(x^2)$. By using the given initial value. Accordingly, we obtain the following successive approximations:

$$u_0(x,t)=e^{x^2},$$
$$u_1(x,t)=e^{x^2}(1+t-\frac{1}{2}t^2+\text{some terms of } x^\alpha t,^\beta\ \alpha\geq 1,\beta\geq 1),$$
$$u_2(x,t)=e^{x^2}(1+t+\frac{1}{2}t^2-\frac{11}{6}t^3+\frac{3}{8}t^4+\text{some terms of } x^\alpha t,^\beta\ \alpha\geq 1,\beta\geq 1),$$
$$u_3(x,t)=e^{x^2}(1+t+\frac{1}{2}t^2+\frac{1}{6}t^3-\frac{119}{24}t^4+\text{some terms of } x^\alpha t,^\beta\ \alpha\geq 1,\beta\geq 1),$$
$$\vdots$$

Recall that

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$

and all terms $x^{\alpha}t^{\beta}$ are neglected whereas $x \to 0$, or $t \to 0$.

Consequently, we find the exact solution is the form:

$$u(x,t) = e^{(x^2+t)}. \qquad (60)$$

From Eqs. (57) and (60), the approximate solution of the given problem 3 by using differential transformation method is the same results as that obtained by the variational iteration method and by the Adomian decomposition method [38] respectively and it clearly appears that the approximate solution remains closed form to exact solution.

Example 4: Finally, we solve problem 4, when $D = 1$ and $p(x,t) = 2t - 2 - 4x^2$.

(i) By using Differential Transformation Method **(DTM)**,

Taking the differential transformation of (13), we have

$$U(k,h+1) = \frac{1}{(h+1)}\left[\begin{array}{l}(k+1)(k+2)U(k+2,h) \\ \quad +2\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,h-1-s)U(k-r,s) \\ \quad -4\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-2,h-s)U(k-r,s)-2U(k,h)\end{array}\right]. \qquad (61)$$

From the initial condition (14), we get

$$U(k,0) = \begin{cases} 0, & \text{if } k = 1,3,5,\cdots \\ \dfrac{1}{(\frac{k}{2})!}, & \text{if } k = 0,2,4,6,\cdots \end{cases} \qquad (62)$$

From the boundary condition (15), consequently we have

$$U(k,0) = \begin{cases} 0, & \text{if } h = 1,3,5,\cdots \\ \dfrac{1}{(\frac{h}{2})!}, & \text{if } h = 0,2,4,6,\cdots \end{cases} \qquad (63)$$

and

$$U(1,h) = 0, \quad \text{forall} \quad h \geq 0. \qquad (64)$$

Using (62)-(64) into (61) by recursive method, we have the following values in Table 3 when $h \geq 1$ we have,

$$\begin{array}{l} U(k,h) = 0, \\ \text{if} \quad k = h = 1,3,5,\ \cdots \\ \text{if} \quad k = 1,3,5,\ \cdots \text{and} \quad h = 2,4,6,\cdots \end{array} \qquad (65)$$

Table 3

$$\begin{array}{lllll}
U(2,2)=1 & U(4,2)=\frac{1}{2} & U(6,2)=\frac{1}{2} & U(8,2)=\frac{1}{24} & \cdots \\
U(2,4)=\frac{1}{2} & U(4,4)=\frac{1}{4} & U(6,4)=\frac{1}{12} & U(8,4)=\frac{1}{48} & \cdots \\
U(2,6)=\frac{1}{6} & U(4,6)=\frac{1}{12} & U(6,6)=\frac{1}{36} & U(8,6)=\frac{1}{144} & \cdots \\
U(2,8)=\frac{1}{24} & U(4,8)=\frac{1}{48} & U(6,8)=\frac{1}{144} & U(8,8)=\frac{1}{576} & \cdots \\
U(2,10)=\frac{1}{120} & U(4,10)=\frac{1}{240} & U(6,10)=\frac{1}{720} & U(8,10)=\frac{1}{2880} & \cdots \\
U(12,2)=\frac{1}{120} & U(10,4)=\frac{1}{240} & U(10,8)=\frac{1}{2880} & U(10,10)=\frac{1}{14400} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

when we substitute all values $U(k,h)$ into (22) as $M\to\infty$ and $N\to\infty$, we obtain series for $u(x,t)$. Then when we rearrange the solution, we get the following closed form solution:

$$u(x,t)=e^{(x^2+t^2)} \tag{66}$$

(ii) By using Variational Iteration Method **(VIM)**,

Using the variational iteration method for solving problem 4, when $D=1$, $p(x,t)=(2t-2-4x^2)$, we have the Lagrange's multiplier $\lambda(s)$ is given by

$$\lambda(s)=s^2-2s-4sx^2-t^2+2t+4tx^2-1. \tag{67}$$

When we substitute the value $\lambda(s)$ of (67) into the functional (34), we get the iteration formula:

$$U_{n+1}(x,t)=u_n(x,t)+\int_0^t(s^2-2s-4sx^2-t^2+2t+4tx^2-1)[\frac{\partial u_n}{\partial s}(x,s)$$
$$-\frac{\partial^2 u_n}{\partial x^2}(x,s)+u_n(x,s)]ds. \tag{68}$$

We can select $u_0(x,t)=\exp(x^2)$. By using the given initial value. Accordingly, we obtain the following successive approximations:

$$u_0(x,t)=e^{x^2},$$
$$u_1(x,t)=e^{x^2}(1+t^2-\frac{2}{3}t^2+\frac{1}{2}t^4+\text{some terms of } x^\alpha t,^\beta\ \alpha\ge 1,\ \beta\ge 1),$$
$$u_2(x,t)=e^{x^2}(1+t^2-\frac{1}{2}t^4+\frac{4}{15}t^5+\frac{1}{30}t^6-\frac{13}{105}t^7+\frac{1}{24}t^8+\text{some terms}$$
$$\text{of } x^\alpha t,^\beta\ \alpha\ge 1,\beta\ge 1),$$
$$u_3(x,t)=e^{x^2}(1+t^2+\frac{1}{2}t^4-2t^5+\frac{119}{90}t^6-\frac{74}{105}t^7+\frac{1}{56}t^8+\frac{4}{945}t^9$$
$$-\frac{197}{37800}t^{10}-\frac{271}{41580}t^{11}+\frac{1}{720}t^{12}+\text{some term of } x^\alpha t,^\beta\ \alpha\ge 1,\ \beta\ge 1),$$
$$\vdots$$

Recall that

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$

and all terms $x^\alpha t^\beta$ are neglected whereas $x\to 0$, or $t\to 0$.

Consequently, we find the exact solution is the form:

$$u(x,t) = e^{(x^2+t^2)}. \tag{69}$$

From Eqs. (66) and (69), the approximate solution of the given problem 3 by using differential transformation method is the same results as that obtained by the variational iteration method and by the Adomian decomposition method [38] respectively and it clearly appears that the approximate solution remains closed form to exact solution.

5. Discussions

The main goal of this work is to conduct a comparative study between the differential transformation method (DTM) and He's variational iteration method (VIM). The two methods are so powerful and efficient that they both give approximations of higher accuracy and closed form solutions if existing. Differential transformation method (DTM) provides the components of the exact solution when these components follow the summation given in (18) and (22). However, He's variational iteration method (VIM) gives several successive approximations through using the iteration of the correction functional. Moreover, the differential transformation method (DTM), which is based on the Taylor series expansion, constructs an analytical solution in the form of polynomial series solution by means of an iterative procedure; whereas He's variational iteration method (VIM) requires the evaluation of the Lagrange multiplier λ. For the examples presented in this paper a closed-form solution is always obtained. In all examples symbolic numerical computations in Maple may need to be performed in general.

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