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# Neighborhood Number in Graphs

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## Abstract

A set $S$ of points in graph $G$ is a neighborhood set if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of $G$ induced by $v$ and all points adjacent to $v$. The *neighborhood number*, denoted $n_o(G)$, of $G$ is the minimum cardinality of a neighborhood set of $G$. In this paper, we study the neighborhood number of certain graphs.

**Keywords**: Neighborhood set; Neighborhood number; Jahangir graph; Harary graphs; Circulant graph.

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## 1. Introduction

In this paper, we concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notations on graphs that are not defined here can be found in [6].

We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. For any vertex $v$ of $G$, the *open neighborhood* of $v$ is the set $N(v, G) = \{u \in V(G): uv \in E(G)\}$, while the *closed neighborhood* of $v$ is the set $N[v, G] = N(v, G) \cup \{v\}$. The degree of $v$ is defined as $\deg(v, G) = |N(v, G)|$. The *maximum* and *minimum degree* of vertices in $V(G)$ are denoted by

$\Delta(G)$ and $\delta(G)$, respectively. If $\Delta(G) = \delta(G) = k$ then $G$ is said to be $k$-regular. We denote the distance between two vertices $x$ and $y$ in $G$ by $d_G(x, y)$. For a set of vertices $S \subseteq V(G)$, $N(S, G)$ is the union of $N(x, G)$ for all $x \in S$, and $N[S, G] = N(S, G) \cup \{S\}$. A cycle on $n$ vertices is denoted by $C_n$.

A set $S$ of vertices in a graph $G$ is a *neighborhood set* (written $n_o$*-set*) of $G$, if $G = \bigcup_{v \in S} \langle N[v] \rangle$ where $\langle N[v] \rangle$ is the subgraph of $G$ induced by $v$ and all vertices adjacent to $v$. The *neighborhood number* $n_o(G)$ of $G$ is the minimum cardinality of $n_o$-sets of $G$. An $n_o$-set of $G$ with cardinality $n_o(G)$ is called $n_o$-set of $G$. We give $n_o$*-critical* if the removal of any vertex from the graph decreases the *close neighborhood number*. For more on concepts of domination and neighborhood critical see [3, 4, 5].

For $m, n \geq 2$, the generalized Jahangir graph $J_{m,n}$ is a graph on $mn + 1$ vertices, i.e., a graph consisting of a cycle $C_{mn}$ with one additional vertex which is adjacent to $n$ vertices of $C_{mn}$ at distance $m$ to each other on $C_{mn}$.

For $2 \leq k < n$, the Harary graph $H_{k,n}$ on $n$ vertices is defined in [9] as follows. Place $n$ vertices around a circle, equally spaced. If $k$ is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\frac{k}{2}$ vertices in each direction around the circle. If $k$ is odd and $n$ is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\frac{k-1}{2}$ vertices in each direction around the circle and to the diametrically opposite vertex. In both cases, $H_{k,n}$ is $k$-regular. If both $k$ and $n$ are odd, $H_{k,n}$ is constructed as follows. It has vertices $0, 1, \ldots, n-1$ and is constructed from $H_{k-1,n}$ by adding edges joining vertex $i$ to vertex $i + \frac{n-1}{2}$ for $0 \leq i \leq \frac{n-1}{2}$.

The circulant graph $C_{n+1}\langle 1, k \rangle$ is the graph with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{v_i v_{i+j \ (mod \ n+1)} | i \in \{0,1, \ldots, n\}\}$ and $j \in \{1, k\}$ where $k \geq 2$ is an integer.
It is necessary for circulant graphs to be connected [2]. Theoretical properties of circulant graphs have been studied extensively and are surveyed [1].

Here we would like to study the neighborhood parameter of these family of graphs.

## 2. Main results

We make the following observations and proposition from [5].

**Observation 2.1.** [5] $n_o(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

**Observation 2.2.** [5] $n_o(K_n) = 1$.

**Proposition 2.3.** [5] For a graph $G$ of order $n$, $n_o(G) = 1$ if and only if $G$ has a point of degree $n - 1$.

The following result can be implied by Proposition 2.3.

**Lemma 2.4.** $n_o(G) = 1$ where $G \in \{H_{m,m+1}, H_{r,r+2}\}$ and $r$ is an odd integer.

**Theorem 2.5.** $n_o(J_{m,n}) = \begin{cases} n_o(C_{mn}) & where\ m\ is\ even \\ n_o(C_{mn}) + 1 & where\ m\ is\ odd \end{cases}.$

**Proof**. Let $\{x_1, x_2, \dots, x_{mn}, u\}$ be the vertex set of $J_{m,n}$. We consider the following two cases:

Case 1: If $m$ is even. Define $S = \{x_1, x_3, \dots, x_{mn-1}\}$. It is easy to check that $S$ is the neighborhood set of $J_{m,n}$ where $m$ is even. It follows that $n_o(J_{m,n}) \le n_o(C_{mn})$. Furthermore, $S$ is $n_o$-set of $C_{mn}$, too. It suffices to note that $u$ cannot be belong to any $n_o$-set of $J_{m,n}$ where $m$ is even. Hence, the desired result completes.

Case 2. If $m$ is odd. Define $S = D \cup \{u\}$ where $D$ is an $n_o$-set of $C_{mn}$. It is not so difficult to check that $u$ is belongs to any $n_o$-set of $J_{m,n}$ where $m$ is odd. Hence the result holds. ■

It is well-known that $H_{2,n} = C_n$. Therefore, we may assume that $m \ge 4$ in the following result.

**Theorem 2.6.** Let $m$ be an even integer. Then $n_o(H_{m,n}) = \left\lceil \frac{n}{\frac{m}{2}+1} \right\rceil$ where $n \ge m + 2$.

**Proof**. Let $n = k\left(\frac{m}{2} + 1\right) + l$ where $k \ge 2$ and $0 \le l \le \frac{m}{2}$. It is easy to check that $S = \{x_1, x_{1+\left(\frac{m}{2}+1\right)}, x_{1+2\left(\frac{m}{2}+1\right)}, \dots, x_{1+(k-1)\left(\frac{m}{2}+1\right)}, x_{1+k(\frac{m}{2}+1)}\}$ is an $n_o$-set of $H_{m,n}$ which implies that $n_o(H_{m,n}) \le \left\lceil \frac{n}{\frac{m}{2}+1} \right\rceil$ (Note that if $l = 0$, then $1 + k(\frac{m}{2} + 1)$ will be modulated to $n$). It now suffices to complete of the proof that $n_o(H_{m,n}) \ge \left\lceil \frac{n}{\frac{m}{2}+1} \right\rceil$.

Without loss of generality, we can assume that $x_i \in S$, where $S$ is an arbitrary $n_o$-set. We show that the longest consecutive vertex $x_j$ of $x_i$ can be in $S$ such that $d_{C_n}(x_i, x_j) \le \frac{m}{2} + 1$. Suppose to the contrary, that $d_{C_n}(x_i, x_j) \ge \frac{m}{2} + 2$. It suffices that, we verify the case $d_{C_n}(x_i, x_j) = \frac{m}{2} + 2$. One can see that there is some edge in $H_{m,n}$ between $x_i$ and $x_j$ which cannot be found in $\langle N[x_i]\rangle \cup \langle N[x_j]\rangle$. Hence the desired result completes. ■

Now, we study the neighborhood number of Harary graphs of odd order. Note that, if $n \in \{m + 1, m + 2\}$, then $n_o(H_{m,n}) = 1$ by Lemma 2.4. Thus, we can assume that $n \ge m + 3$ in the following result.

**Theorem 2.7.** *Let $m$ be an odd integer and let $n = 2k$ or $n = 2k + 1$ for every $k$. Then*

$$n_o(H_{m,n}) = \begin{cases} k & where\ n\ is\ even\ and\ k\ is\ odd \\ k + 1 & where\ n\ is\ odd\ or\ n\ and\ k\ are\ even \end{cases}.$$

**Proof.** Let $n = 2k$ *or* $n = 2k + 1$. Since $m$ is odd then we have $\left\lceil \frac{n}{2} \right\rceil$ diameters in $H_{m,n}$. Moreover, by the structure of the graph, it follows that $n_o(H_{m,n}) \geq k$ where $n$ is even and $k$ is an odd integer and $n_o(H_{m,n}) \geq k + 1$ where $n$ is odd or $n$ and $k$ are even.

It now suffices to complete of the proof that we assign an $n_o(G)$-set. Let $\{x_1, x_3, x_5, \dots, x_{n-3}, x_{n-1}\}$ where $n$ is even and $k$ is odd. Let $\{x_1, x_3, x_5, \dots, x_{n-2}, x_n\}$ where $n$ is odd and $k$ is even. Let $\left\{x_1, x_3, x_5, \dots, x_{\left\lceil\frac{n}{2}\right\rceil-1}, x_{\left\lceil\frac{n}{2}\right\rceil+1}, x_{\left\lceil\frac{n}{2}\right\rceil+2}, x_{\left\lceil\frac{n}{2}\right\rceil+4}, \dots, x_{n-1}\right\}$ where $n$ and $k$ are odd. Let $\{x_1, x_3, \dots, x_{\frac{n}{2}-3}, x_{\frac{n}{2}-1}, x_{\frac{n}{2}}, x_{\frac{n}{2}+2}, x_{\frac{n}{2}+4}, \dots, x_{n-2}, x_{n-1}\}$ where $n$ and $k$ are even. Hence, by these assumptions, the desired result completes. ■

Now, we study the neighborhood number of circulant graphs. Note that, if $m = 2$, then $C_{n+1}\langle 1,2\rangle = H_{4,n}$ which verified in Theorem 2.6. From this, we may assume that $m \geq 3$ in the following result.

**Theorem 2.8.** *Let* $G = C_{n+1}\langle 1, m\rangle$ *where* $m$ *and* $n$ *are odd integers. Then* $n_o(G) = n_o(C_{n+1})$.

**Proof**. Let $G = C_{n+1}\langle 1, m\rangle$ where $m$ and $n$ are odd integers. It is clear to see that $n_o(G) \geq$ *no(Cn+1)*. It suffices to complete of the proof that *{x1,x3, x5,…,xn}* is an *no*-set of *G*. ■

Assume that $C_i$ and $C_j$ are two $m + 1$-cycles of circulante graph $C_{n+1}\langle 1, m\rangle$, we say these two cycles are independent, that is $V(C_i) \cap V(C_j) = \emptyset$.

**Theorem 2.9.** *Let* $G = C_{n+1}\langle 1, m\rangle$ *where* $m$ *and* $n + 1$ *are odd integers such that* $n + 1 = k(m + 1) + l$ for every odd $l$ with $1 \leq l \leq m$. *Then* $n_o(G) = k.n_o(C_{m+1}) + \frac{m+l}{2}$.

**Proof**. Let $n + 1 = k(m + 1) + l$ for every odd $l$ with $1 \leq l \leq m$. Suppose that $\cup_{i=1}^{k} \{x_1^i, x_2^i, x_3^i, \dots, x_{m+1}^i\} \cup \cup_{j=1}^{l} \{y_j\}$ and $S$ are the vertex set and $n_o$-set of $C_{n+1}\langle 1, m\rangle$, respectively. It is easy to see that, we have $k$ independent $m + 1$-cycle. Suppose that $C_{m+1}^i: x_1^i, x_2^i, \dots, x_{m+1}^i$ is an $m + 1$-cycle. Certainly, $|S \cap C_{m+1}^i| \geq n_o(C_{m+1})$ for each $1 \leq i \leq k$. It follows that $n_o(C_{n+1}\langle 1, m\rangle) \geq k.n_o(C_{m+1})$. Now, let $D \subseteq S$, for which $S$ and $D$ are $n_o$-set of $G$ and $k$ independent $m + 1$-cycle in H=$\bigcup_{v\in D}\langle N[v]\rangle$, respectively. It is not difficult to see that, we have $m + 1$ vertices in $H$ such that $m - 1$ vertices of degree 3, $l - 1$ vertices of degree 1 and two vertices of degree 2. By these assumptions, it follows that $n_o(C_{n+1}\langle 1, m\rangle) \geq k.n_o(C_{m+1}) + \frac{m+l}{2}$ where $m$ and $n + 1$ are odd integers.

It suffices to complete of the proof that, $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \dots, x_m^i\} \cup \{x_{m+1}^k, x_{m-1}^k, x_{m-3}^k, \dots, x_{m-t}^k\} \cup \cup_{j=1}^{l} \{y_j\}$ is an $n_o$-set of graph for which $l$ is an odd integer, $1 \leq l < m$ and $t = m - l - 3$. Finally, let $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \dots, x_m^i\} \cup \cup_{j=1}^{l} \{y_j\}$ where $l = m$. ■

**Theorem 2.10.** *Let* $G = C_{n+1}\langle 1, m\rangle$ *where* $m$ *is an even integer. Then*

$$n_o(G)=\begin{cases} k.n_o(C_{m+1})+l & \text{for every even } l\leq m-2\\ k.n_o(C_{m+1})+l-1 & \text{for } l=m\\ k.n_o(C_{m+1})+\left[\frac{m-1}{2}\right] & \text{for every odd } l\leq m-3\\ k.n_o(C_{m+1})+\frac{m}{2} & \text{for } l=m-1\end{cases}$$

**Proof**. Let $n+1=k(m+1)+l$ where $0\leq l\leq m$. Suppose that $\cup_{i=1}^{k}\{x_1^i,x_2^i,x_3^i,\dots,x_{m+1}^i\}\cup\cup_{j=1}^{l}\{y_j\}$ and $S$ are the vertex set and $n_o$-set of $C_{n+1}\langle 1,m\rangle$, respectively. (note that if $l=0$, then $Y=\cup_{j=1}^{l}\{y_j\}=\emptyset$). It is easy to see that we have $k$ independent $m+1$-cycle. Suppose that $C_{m+1}^i$: $x_1^i,x_2^i,\dots,x_{m+1}^i$ is an $m+1$-cycle. Certainly, $|S\cap C_{m+1}^i|\geq n_o(C_{m+1})$ for each $1\leq i\leq k$. It follows that $n_o(C_{n+1}\langle 1,m\rangle)\geq k.n_o(C_{m+1})$. Now, let $D\subseteq S$, for which $S$ and $D$ are $n_o$-set of $G$ and $k$ independent $m+1$-cycle in H=$\cup_{v\in D}\langle N[v]\rangle$, respectively. It is not difficult to see that:

(i) For every even $l\leq m-2$, we have $2l+1$ vertices in $H$ such that $l+1$ vertices of degree 3, $l-1$ vertices of degree 1 and one vertex of degree 2. By these assumptions, it follows that $n_o(G)\geq k.n_o(C_{m+1})+l$.

(ii) For $l=m$, we have $2l-1$ vertices in $H$ such that $l-2$ vertices of degree 3, $l-2$ vertices of degree 1 and three vertices of degree 2. By these assumptions, it follows that $n_o(G)\geq k.n_o(C_{m+1})+l-1$ where $l=m$.

(iii) For every odd $l\leq m-3$, there exist $\frac{l-1}{2}$ vertices in $V(G)$ such that these do not belong to $V(H)$. Further all of them must be in $n_o(G)$-set. Also, we have $m+l-2-\frac{l-1}{2}=m-1+\frac{l-1}{2}$ vertices in $H$ such that $m-2$ vertices of degree 3, $\frac{l-1}{2}+1$ vertices of degree 2 and each of these vertices, except two of them, are between a pair of those $\frac{l-1}{2}$ vertices which are not in $H$. Hence, $\frac{l-1}{2}$ vertices out of $H$ can cover at most $2l-2$ vertices which are not of degree 4 in $H$. Meanwhile, we have $m+l-2-4\left(\frac{l-1}{2}\right)=m-l$. By these assumptions, it follows that $n_o(G)\geq k.n_o(C_{m+1})+[\frac{m-1}{2}]$.

(iv) For $l=m-1$, there exist $\frac{l-1}{2}$ vertices in $V(G)$ such that these do not belong to $V(H)$. Further all of them must be in $n_o(G)$-set. Also, we have $m+l-1-\frac{l-1}{2}=m+\frac{l-1}{2}$ vertices in $H$ such that $m$ vertices of degree 3, $\frac{l-1}{2}$ vertices of degree 2 and each of these vertices, except two of them, are between a pair of those $\frac{l-1}{2}$ vertices which are not in $H$. Hence, $\frac{l-1}{2}$ vertices out of $H$ can cover at most $2l-2$ vertices which are not of degree 4 in $H$. Meanwhile, we have $m+l-1-4\left(\frac{l-1}{2}\right)=m-l+1=2$. By these assumptions, it follows that $n_o(G)\geq k.n_o(C_{m+1})+\frac{l-1}{2}+1=k.n_o(C_{m+1})+\frac{m}{2}$.

It suffices to complete of each part of the proof that, we assign an $n_o$-set for the graph:

(i) For every even $l < m$, consider $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \ldots, x_m^i\} \cup \{x_{m+1}^k\} \cup Y - \{y_l\}$.

(ii) For $l = m$, consider $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \ldots, x_m^i\} \cup Y - \{y_{l-1}\}$.

(iii) For every odd $l \leq m - 3$, let $t = m - l - 3$. If $t = 0$, consider $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \ldots, x_m^i\} \cup \{y_1, y_3, x_5, \ldots, y_l\}$ and for $t \geq 1$, consider $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \ldots, x_m^i\} \cup \{x_{m-2}^k, x_{m-4}^k, \ldots, x_{m-t}^k\} \cup \{y_1, y_3, y_5, \ldots, y_l\}$.

(iv) For $l = m - 1$, consider $S = \cup_{i=1}^{k} \{x_1^i, x_3^i, \ldots, x_m^i\} \cup \{y_1, y_3, \ldots, y_l\}$. ■

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