Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 17 (2017), 345–354
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Strong convergence of modified viscosity implicit approximation methods for asymptotically nonexpansive mappings in complete CAT(0) spaces

Nuttapol Pakkaranang[a], Poom Kumam[a,b,c,*], Yeol Je Cho[d,e], Plern Saipara[a], Anantachai Padcharoen[a], Chatuphol Khaofong[a]

[a]*Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.*

[b]*Department of Mathematics & Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand.*

[c]*Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan.*

[d]*Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Korea.*

[e]*Center for General Education, China Medical University, Taichung, 40402, Taiwan.*

Abstract

In this paper, we introduce a modified viscosity implicit iteration for asymptotically nonexpansive mappings in complete CAT(0) spaces. Under suitable conditions, we prove some strong convergence to a fixed point of an asymptotically nonexpansive mapping in a such space which is also the solution of variational inequality. Moreover, we illustrate some numerical example of our main results. Our results extend and improve some recent result of Yao et al. [Y.-H. Yao, N. Shahzad, Y.-C. Liou, Fixed Point Theory Appl., **2015** (2015), 15 pages] and Xu et al. [H.-K. Xu, M. A. Alghamdi, N. Shahzad, Fixed Point Theory Appl., **2015** (2015), 12 pages].

Keywords: Asymptotically nonexpansive mapping, projection, viscosity, implicit iterative rule, variational inequality, CAT(0) spaces.
2010 MSC: 47H09, 47H10, 47J25.

1. Introduction

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that

$$c(0) = x, \qquad c(r) = y, \qquad d(c(t), c(s)) = |t - s|$$

for all $s, t \in [0, r]$. In particular, c is an isometry and $d(x, y) = r$. The image of c is called a *geodesic* (*or metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. We denote the point $w \in [x, y]$ such that $d(x, w) = \alpha d(x, y)$ by $w = (1 - \alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$.

*Corresponding author
Email addresses: nuttapol.pakka@mail.kmutt.ac.th (Nuttapol Pakkaranang), poom.kum@kmutt.ac.th (Poom Kumam), yjcho@gnu.ac.kr (Yeol Je Cho), splernn@gmail.com (Plern Saipara), apadcharoen@yahoo.com (Anantachai Padcharoen), chatuphol.k@hotmail.com (Chatuphol Khaofong)

doi:10.22436/jmcs.017.03.01

Received 2017-02-08

The space (X, d) is called a *geodesic space* if any two points of X are joined by a geodesic and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $D \subseteq X$ is said to be *convex* if D includes geodesic segment joining every two points of itself. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for geodesic triangle (or $\triangle(x_1, x_2, x_3)$) in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. A geodesic metric space is called a CAT(0) space ([5]) if all geodesic triangles satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if, for all $x, y \in \triangle$ and all comparison points,

$$d(x, y) \leqslant d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which is denoted by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leqslant \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{1.1}$$

The inequality (1.1) is called the (CN) *inequality* (for more details, see Bruhat and Titz [6]). In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well-known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings, Pre-Hilbert spaces, $\mathbb{R}$-trees ([5]), the complex Hilbert ball with a hyperbolic metric is a CAT(0) space. Further, Complete CAT(0) spaces are called *Hadamard* spaces.

Let C be a nonempty subset of a complete CAT(0) space X. Then a mapping $T : C \to C$ is called

(1) nonexpansive if and only if $d(Tx, Ty) \leqslant d(x, y), \quad \forall x, y \in C$;

(2) asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leqslant k_n d(x, y), \ \forall n \geqslant 1, \ x, y \in C$.

A point $x \in X$ is called a fixed point of T if $x = Tx$. We will denote by $F(T)$ the set of fixed points of T. Kirk [12] proved the existence theorem of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces.

A mapping $f : C \to C$ is called a contraction with coefficient $k \in [0, 1)$ if and only if

$$d(f(x), f(y)) \leqslant kd(x, y), \qquad \forall x, y \in C.$$

f has a unique fixed point when C is a nonempty, closed, and subset of a complete metric space was guaranteed by Banach's contraction principle [3]. The existence theorems of fixed points and convergence theorems for various mappings in CAT(0) spaces have been investigated by many authors [7–10, 13]. Obviously every contraction mapping is nonexpansive and every nonexpansive mapping is asymptotically nonexpansive with the sequence $\{k_n = 1\}$ for all $n \geqslant 1$.

One of the powerful numerical methods is implicit midpoint method for solving ordinary differential equations and differential algebraic equations. For a study discussion related to this numerical method we refer to [1, 2, 14, 15].

Very recently, Xu et al. [18] introduced the following viscosity implicit midpoint method for a nonexpansive mapping $T : H \to H$ in a Hilbert space H:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right)$$

for all $n \geqslant 0$, where $\alpha_n \in (0,1)$ and $f : H \to H$ is a contraction. Under suitable conditions, they proved that the sequence $\{x_n\}$ converges strongly to a fixed point of a nonexpansive mapping, which is also a unique solution of the following variational inequality

$$\langle (I-f)x^*, x-x^* \rangle \geqslant 0, \quad \forall x \in F(T).$$

Motivated and inspired by Xu et al. [18], we study and introduce the following modefied viscosity implicit iteration for an asymptotically nonexpansive mapping in CAT(0) space X:

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}) \tag{1.2}$$

for all $n \geqslant 0$, where $\alpha_n, \beta_n \in (0,1)$, $T^n : X \to X$ be an asymptotically nonexpansive mapping and $f : X \to X$ is a contraction.

The purpose of this paper, we prove strong convergence theorems of the modified viscosity implicit iteration process for an asymptotically nonexpansive mapping in CAT(0) space under suitable conditions. We also show that the limit of the sequence $\{x_n\}$ generated by (1.2) solves the solution of the following variational inequality

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geqslant 0, \quad \forall p \in F(T).$$

Furthermore, we illustrate some numerical examples for support our main results.

2. Preliminaries

In this section, we always suppose that X is a CAT(0) space and write $(1-t)x \oplus ty$ for the unique point w in the geodesic segment joining from x to y which is, $[x,y] = \{(1-\lambda)x \oplus \lambda y : \lambda \in [0,1]\}$,

$$d(w,x) = \lambda d(x,y), \qquad \text{and} \qquad d(w,y) = (1-\lambda)d(x,y).$$

Lemma 2.1 ([11]). *Let K be a CAT(0) space. For all $x,y,z \in X$ and $\lambda, \gamma \in [0,1]$, we have the followings:*

(i) $d(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda d(x,z) + (1-\lambda)d(y,z)$;
(ii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$;
(iii) $d(\lambda x \oplus (1-\lambda)y, \gamma x \oplus (1-\gamma)y) = |\lambda-\gamma| d(x,y)$;
(iv) $d(\lambda x \oplus (1-\lambda)y, tu \oplus (1-\lambda)w) \leqslant \lambda d(x,u) + (1-\lambda)d(y,w)$.

Lemma 2.2 ([17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \xi_n$$

for all $n \geqslant 0$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\xi_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \dfrac{\xi_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\xi_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The concept of quasilinearization in X was introduced by Berg and Nikolaev [4].

Denote a pair $(p,q) \in X \times X$ by $\overrightarrow{pq}$ and call it a vector. Then quasi-linearization is defined as a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ such that

$$\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = \frac{1}{2}(d^2(p,s) + d^2(q,r) - d^2(p,r) - d^2(q,s))$$

for all $p,q,r,s \in X$.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. The metric projection $P_C : X \to C$ is defined by

$$w = P_c(x) \iff d(w,x) = \inf\{d(y,x) : y \in C\}$$

for all $x \in X$.

Lemma 2.3 ([4]). *Let C be a nonempty closed and convex subset of a complete CAT(0) space X, $x \in X$ and $w \in C$. Then $w = P_c(x)$ if and only if*

$$\langle \overrightarrow{yw}, \overrightarrow{wx} \rangle \geqslant 0$$

for all $y \in C$.

Lemma 2.4 ([16]). *Let X be a CAT(0) space, C be a nonempty closed and convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$. For any contraction $f : C \to C$ and $\lambda \in (0, 1)$, let $x_\lambda \in C$ be the unique fixed point of the contraction $x \mapsto \lambda f(x) \oplus (1-\lambda)Tx$, i.e.,*

$$x_\lambda = \lambda f(x_\lambda) \oplus (1-\lambda)T^n x_\lambda.$$

Then $\{x_n\}$ converges strongly as $\lambda \to 0$ to a point $\tilde{p}$ such that

$$\tilde{p} = P_{F(T)} f(\tilde{p}),$$

which is the unique solution to the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geqslant 0$$

for all $p \in F(T)$.

3. Main results

Now, we prove the main results as follows.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d) and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_1 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}) \tag{3.1}$$

for all $n \geqslant 0$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in interval $(0, 1)$, satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$;
(iv) $\dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} \to 0$, *as* $n \to \infty$;
(v) T *satisfies the asymptotically regular* $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)} f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geqslant 0, \quad \forall p \in F(T). \tag{3.2}$$

Proof. We divide the proof into three steps.

Step 1. First, we prove that for all $v \in C$, the mapping defined by

$$x \to T_v(x) := \alpha f(v) \oplus (1-\alpha)T^n(\beta v \oplus (1-\beta)x)$$

for all $x \in C$, $\alpha, \beta \in (0, 1)$ and f is a contraction mapping with the contractive constant $(1-\alpha)k_n(1-\beta)$. Indeed, it follows from Lemma 2.1 that, for all $x, y \in C$

$$d(T_v x, T_v y) = d(\alpha f(v) \oplus (1-\alpha)T^n(\beta v \oplus (1-\beta)x), \alpha f(v) \oplus (1-\alpha)T^n(\beta v \oplus (1-\beta)y))$$

$$\begin{aligned}
&\leqslant \alpha d(f(\nu), f(\nu)) + (1-\alpha) d(T^n(\beta\nu \oplus (1-\beta)x), T^n(\beta\nu \oplus (1-\beta)y)) \\
&\leqslant (1-\alpha)k_n(\beta d(\nu,\nu) + (1-\beta)d(x,y)) \\
&\leqslant (1-\alpha)k_n(1-\beta)d(x,y).
\end{aligned}$$

It follows that $0 < (1-\alpha)k_n(1-\beta) < 1$. This implies that the mapping $T_\nu : C \to C$ is contraction with a constant $(1-\alpha)k_n(1-\beta)$. Thus the sequence $\{x_n\}$ defined by (3.1) is well-defined.

Step 2. Next, we prove that the sequence $\{x_n\}$ is bounded. By condition (iii), for any $0 < \varepsilon < 1-\alpha$ sufficient large $n \geqslant 0$, we have $k_n - 1 \leqslant \alpha_n \varepsilon$. For all $p \in F(T)$, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p) \\
&\leqslant \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), p) \\
&\leqslant \alpha_n (d(f(x_n), f(p)) + d(f(p), p)) + (1-\alpha_n)k_n(\beta_n d(x_n, p) + (1-\beta_n)d(x_{n+1}, p)) \\
&\leqslant \alpha_n k d(x_n, p) + \alpha_n d(f(p), p) + k_n(1-\alpha_n)\beta_n d(x_n, p) + k_n(1-\alpha_n)(1-\beta_n)d(x_{n+1}, p),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{n+1}, p) &\leqslant \frac{\alpha_n k + \beta_n k_n - \alpha_n\beta_n k_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(x_n, p) + \frac{\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(f(p), p) \\
&= (1 - \frac{(1-k_n+\alpha_n k_n - \alpha_n k)}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, p) + \frac{\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(f(p), p) \\
&= (1 - \frac{-(k_n - 1 - \alpha_n k_n + \alpha_n k_n)}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(x_n, p) + \frac{\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(f(p), p) \\
&\leqslant (1 - \frac{-(\alpha_n\varepsilon - \alpha_n k_n + \alpha_n k)}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, p) + \frac{\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(f(p), p) \\
&= (1 - \frac{(k_n - k - \varepsilon)\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, p) \\
&\quad + \frac{(k_n - k - \varepsilon)\alpha_n}{(1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n)(k_n - k - \varepsilon)} d(f(p), p) \\
&\leqslant (1 - \frac{(1-k-\varepsilon)\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n}) d(x_n, p) + \frac{(1-k-\varepsilon)\alpha_n}{(\alpha_n+\beta_n-\alpha_n\beta_n)(1-k-\varepsilon)} d(f(p), p) \\
&\leqslant \max\{d(x_n, p), \frac{d(f(p), p)}{(1-k-\varepsilon)}\}.
\end{aligned}$$

By mathematical induction, we can prove that

$$d(x_n, p) \leqslant \max\{d(x_0, p), \frac{d(f(p), p)}{(1-k-\varepsilon)}\}$$

for all $n \geqslant 0$. This implies that the sequence $\{x_n\}$ is bounded, so $\{f(x_n)\}$ and $\{T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1})\}$ are also bounded.

Step 3. Next, we prove that the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)}f(\tilde{p})$. Set

$$w_n = \alpha_n f(w_n) \oplus (1-\alpha_n)T^n(w_n)$$

for all $n \geqslant 0$. By Lemma 2.4, the sequence $\{w_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{p} = P_{F(T)}f(\tilde{p})$, which is the unique solution to the variational inequality (3.2).

On the other hand, it follows from (3.1) and Lemma 2.1, that

$$\begin{aligned}
d(x_{n+1}, w_n) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), \alpha_n f(w_n) \oplus (1-\alpha_n)T^n w_n) \\
&\leqslant \alpha_n d(f(x_n), f(w_n)) + (1-\alpha_n) d(T^n(\beta_n x_n \oplus (1-\beta_n)x_{n+1}), T^n w_n) \\
&\leqslant \alpha_n k d(x_n, w_n) + (1-\alpha_n)k_n(\beta_n d(x_n, w_n) + (1-\beta_n)d(x_{n+1}, w_n)),
\end{aligned}$$

which implies that

$$
\begin{aligned}
d(x_{n+1}, w_n) &\leqslant \frac{\alpha_n k + \beta_n k_n - \alpha_n \beta_n k_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n} d(x_n, w_n) \\
&= (1 - \frac{-(k_n - 1 - \alpha_n k_n + \alpha_n k)}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, w_n) \\
&\leqslant (1 - \frac{-(\alpha_n \varepsilon - \alpha_n k_n + \alpha_n k)}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, w_n) \\
&= (1 - \frac{(k_n - k - \varepsilon)\alpha_n}{1-(1-\alpha_n-\beta_n+\alpha_n\beta_n)k_n}) d(x_n, w_n) \\
&\leqslant (1 - \frac{(1 - k - \varepsilon)\alpha_n}{\alpha_n + \beta_n - \alpha_n\beta_n}) d(x_n, w_n) \\
&\leqslant (1 - (1-k-\varepsilon)\alpha_n)(d(x_n, w_{n-1}) + d(w_{n-1}, w_n)) \\
&\leqslant (1 - (1-k-\varepsilon)\alpha_n) d(x_n, w_{n-1}) + d(w_{n-1}, w_n).
\end{aligned}
\tag{3.3}
$$

In order to use Lemma 2.2, it should be proved that

$$
\limsup_{n\to\infty} \frac{d(w_{n-1}, w_n)}{(1-k-\varepsilon)\alpha_n} \leqslant 0. \tag{3.4}
$$

In fact, by Lemma 2.1, we have

$$
\begin{aligned}
d(w_n, w_{n-1}) &= d(\alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_n, \alpha_{n-1} f(w_{n-1}) \oplus (1-\alpha_{n-1}) T^n w_{n-1}) \\
&\leqslant d(\alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_n, \alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_{n-1}) \\
&\quad + d(\alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_{n-1}, \alpha_n f(w_{n-1}) \oplus (1-\alpha_n) T^n w_{n-1}) \\
&\quad + d(\alpha_n f(w_{n-1}) \oplus (1-\alpha_n) T^n w_{n-1}, \alpha_{n-1} f(w_{n-1}) \oplus (1-\alpha_{n-1}) T^n w_{n-1}) \\
&\leqslant (1-\alpha_n) d(T^n w_n, T^n w_{n-1}) + \alpha_n d(f(w_n), f(w_{n-1})) + |\alpha_n - \alpha_{n-1}| d(f(w_{n-1}), T^n w_{n-1}) \\
&\leqslant (1-\alpha_n) k_n d(w_n, w_{n-1}) + \alpha_n k d(w_n, w_{n-1}) + |\alpha_n - \alpha_{n-1}| M^*,
\end{aligned}
$$

where $M^* = \sup_{n\geqslant 1} d(f(w_{n-1}), T^n w_{n-1})$, which implies that

$$
\begin{aligned}
d(w_n, w_{n-1}) &\leqslant \frac{1}{1 - k_n + \alpha_n k_n - \alpha_n k} |\alpha_n - \alpha_{n-1}| M^* \\
&= \frac{1}{-(k_n - 1 - \alpha_n k_n + \alpha_n k)} |\alpha_n - \alpha_{n-1}| M^* \\
&\leqslant \frac{1}{-(\varepsilon - k_n + k)\alpha_n} |\alpha_n - \alpha_{n-1}| M^* \\
&= \frac{1}{(k_n - k - \varepsilon)\alpha_n} |\alpha_n - \alpha_{n-1}| M^* \\
&\leqslant \frac{1}{(1 - k - \varepsilon)\alpha_n} |\alpha_n - \alpha_{n-1}| M^*.
\end{aligned}
$$

By the condition (iv), we have

$$
\limsup_{n\to\infty} \frac{d(w_{n-1}, w_n)}{(1-k-\varepsilon)\alpha_n} \leqslant \limsup_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{(1-k-\varepsilon)^2 \alpha_n^2} M^* = 0.
$$

Thus (3.4) is proved. By Lemma 2.2 and (3.3), we obtain

$$
d(x_{n+1}, w_n) \to 0, \quad \text{as} \quad n \to \infty.
$$

Since $w_n \to \tilde{p} = P_{F(T)} f(\tilde{p})$ and $\tilde{p}$ is the unique solution of the variational inequality (3.2), we have $x_n \to \tilde{p}$ as $n \to \infty$ and $\tilde{p}$ is also the unique solution of variational inequality (3.2). This completes the proof. □

Remark 3.2. Since every Hilbert space is a complete CAT(0) space, Theorem 3.1 generalizes and improves the main results in Yao et al. [19] and Xu et al. [18].

Corollary 3.3. *Let C be nonempty closed and convex subset of a real Hilbert space H and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n(\beta_n x_n + (1 - \beta_n)x_{n+1})$$

for all $n \geqslant 0$, where $\alpha_n, \beta_n \in (0, 1)$ satisfy the conditions (i)-(v) *as in Theorem* 3.1*. Then the sequence $\{x_n\}$ converges strongly to $\tilde{p}$ such that*

$$\tilde{p} = P_{F(T)}f(\tilde{p}),$$

which is the unique solution of the following variational inequality

$$\langle \tilde{p} - f(\tilde{p}), p - \tilde{p}\rangle \geqslant 0, \quad \forall p \in F(T). \tag{3.5}$$

From Theorem 3.1, if sequence $\{k_n := 1\}$, then $T : C \to C$ in Theorem 3.1 is a nonexpansive mapping, we can obtain the following results immediately.

Corollary 3.4. *Let C be nonempty closed and convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(\beta_n x_n \oplus (1 - \beta_n)x_{n+1})$$

for all $n \geqslant 0$, where $\alpha_n, \beta_n \in (0, 1)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} \to 0$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\tilde{p} = P_{F(T)}f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the variational inequality (3.2).

Since every Hilbert space is a complete CAT(0) space, from Corollary 3.4 we can obtain the following result immediately.

Corollary 3.5. *Let C be a nonempty closed and convex subset of real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$ and, for arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)x_{n+1})$$

for all $n \geqslant 0$, where $\alpha_n, \beta_n \in (0, 1)$ and satisfy the conditions (i)-(iii) *as in Corollary* 3.4*. Then the sequence $\{x_n\}$ converges strongly to $\tilde{p}$ such that $\tilde{p} = P_{F(T)}f(\tilde{p})$, which is a fixed point of T and, also, it is a solution of the variational inequality* (3.5)*.*

4. Numerical example

In this section, we will illustrate reckoning the convergence behavior of modified viscosity implicit iteration process (3.1) with numerical results for supporting main theorem.

Example 4.1. Let $X = \mathbb{R}$ be a Euclidean metric space, which is also a complete CAT(0) space and $C = [1, 4]$. Let $T : C \to C$ be defined by

$$Tx = \sqrt{x}.$$

It is obvious that T is an asymptotically nonexpansive mapping and a sequence $\{k_n = 1\}$ with $F(T) = \{1\}$.

Let $f : C \to C$ be defined by

$$fx = \sqrt{x}.$$

It is easy to see that f is contraction mapping.

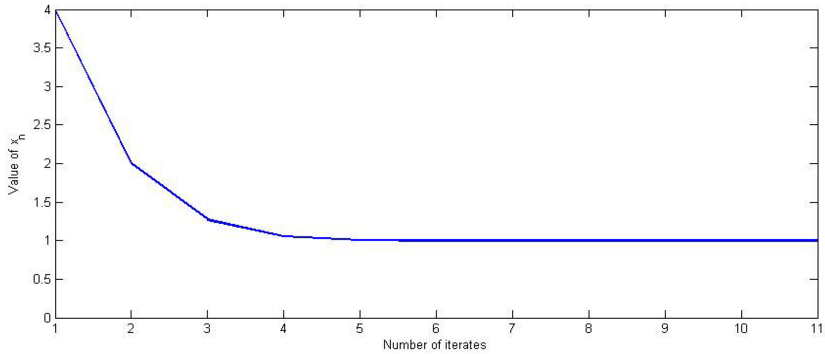

Figure 1: The value of x_n plotting in Table 1.

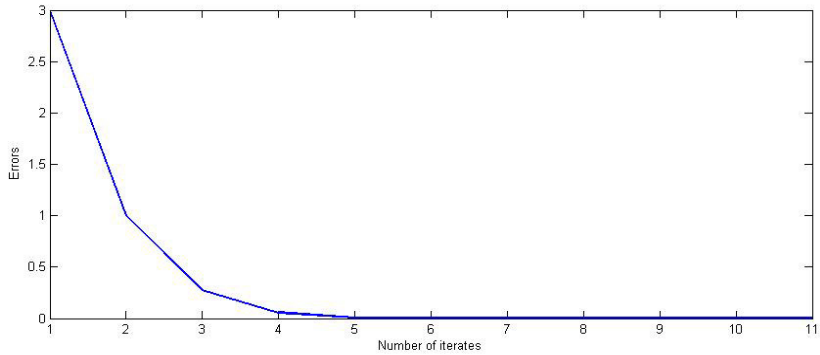

Figure 2: The value of $|x_n - p|$ (error) plotting in Table 1.

Table 1: The values of the sequence $\{x_n\}$ and the error values.

number of iterates	x_n	$\|x_n - p\|$
1	4.000000000000	3.000000000000
2	2.000000000000	1.000000000000
3	1.272611350577	0.272611350577
4	1.055479646443	0.055479646443
5	1.008314617547	0.008314617547
6	1.000945233243	0.000945233243
7	1.000092259589	0.000092259589
8	1.000007257835	7.2578352E-06
9	1.000000479959	4.7995897E-07
10	1.000000028340	2.8339588E-08
11	1.000000001469	1.4694320E-09

Let $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{2}$ for all $n \geqslant 1$ and initial point $x_1 = 4$. Let $p = 1$ and $p \in F(T)$. Then we get numerical results in Table 1. Moreover, we also illustrate the convergence behavior of modified viscosity implicit iteration process (3.1) by the values of x_n as is shown in Figure 1. Figure 2 shows numerical results with an error (10^{-9}).

Next, we will show that the fixed point of T is solution of the following variational inequality:

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geqslant 0, \quad \forall p \in F(T),$$

where $\tilde{p} = P_{F(T)}f(\tilde{p})$.

Proof. Let $p \in F(T)$, and since $F(T) = \{1\}$ hence $p = 1$. By Theorem 3.1, we get

$$p = \tilde{p} = P_{F(T)}f(\tilde{p}).$$

For variational inequality, we have

$$\langle \overrightarrow{\tilde{p}f(\tilde{p})}, \overrightarrow{p\tilde{p}} \rangle \geqslant 0, \quad \langle (1) \cdot f(1), (1) \cdot (1) \rangle \geqslant 0, \quad \langle (1) \cdot (1), (1) \cdot (1) \rangle \geqslant 0.$$

Therefore the fixed point of T is a solution that satisfies the variational inequality. □

Acknowledgment

The authors are very thankful to the anonymous reviewers for their careful reading of the manuscript and for their constructive reports, which have been very useful to improve the paper. This project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Cluster (CLASSIC), Faculty of Science, KMUTT. The first author would like to thank the Research Professional Development Project Under the Science Achievement Scholarship of Thailand (SAST) for financial support. Furthermore, this work was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU59 Grant No.59000399).

References

[1] W. Auzinger, R. Frank, *Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case*, Numer. Math., **56** (1989), 469–499. 1

[2] G. Bader, P. Deuflhard, *A semi-implicit mid-point rule for stiff systems of ordinary differential equations*, Numer. Math., **41** (1983), 373–398. 1

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1

[4] I. D. Berg, I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata, **133** (2008), 195–218. 2, 2.3

[5] M. R. Bridon, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, (1999). 1, 1

[6] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local*, (French) Inst. Hautes tudes Sci. Publ. Math., **41** (1972), 5–251. 1

[7] Y. J. Cho, L. Ćirić, S.-H. Wang, *Convergence theorems for nonexpansive semigroups in* CAT(0) *spaces*, Nonlinear Anal., **74** (2011), 6050–6059. 1

[8] S. Dhompongsa, A. Kaewkhao, B. Panyanak, *Lim's theorems for multivalued mappings in CAT(0) spaces*, J. Math. Anal. Appl., **312** (2005), 478–487.

[9] S. Dhompongsa, W. A. Kirk, B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces*, J. Nonlinear Convex Anal., **8** (2007), 35–45.

[10] S. Dhompongsa, W. A. Kirk, B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal., **65** (2006), 762–772. 1

[11] S. Dhompongsa, B. Panyanak, *On Δ-convergence theorems in* CAT(0) *spaces*, Comput. Math. Appl., **56** (2008), 195–218. 2.1

[12] W. A. Kirk, *Geodesic geometry and fixed point theory*, II, International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, (2004), 113–142. 1

[13] S. Saejung, *Halpern's iteration in CAT(0) spaces*, Fixed Point Theory Appl., **2010** (2010), 13 pages. 1

[14] C. Schneider, *Analysis of the linearly implicit mid-point rule for differential-algebraic equations*, Electron. Trans. Numer. Anal., **1** (1993), 1–10. 1

[15] S. Somalia, *Implicit midpoint rule to the nonlinear degenerate boundary value problems*, Int. J. Comput. Math., **79** (2002), 327–332. 1

[16] R. Wangkeeree, U. Boonkong, P. Preechasilp, *Viscosity approximation methods for asymptotically nonexpansive mapping in* CAT(0) *spaces*, Fixed point Theory Appl., **2015** (2015), 15 pages. 2.4

[17] H.-K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240–256. 2.2

[18] H.-K. Xu, M. A. Alghamdi, N. Shahzad, *The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory Appl., **2015** (2015), 12 pages. 1, 1, 3.2

[19] Y.-H. Yao, N. Shahzad, Y.-C. Liou, *Modified semi-implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory Appl., **2015** (2015), 15 pages. 3.2