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Numerical Solution of Fredholm and Volterra Integral Equations of the First Kind Using Wavelets bases

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Abstract

The Fredholm and Volterra types of integral equations are appeared in many engineering fields. In this paper, we suggest a method for solving Fredholm and Volterra integral equations of the first kind based on the wavelet bases. The Haar, continuous Legendre, CAS, Chebyshev wavelets of the first kind (CFK) and of the second kind (CSK) are used on [0,1] and are utilized as a basis in Galerkin or collocation method to approximate the solution of the integral equations. In this case, the integral equation converts to the system of linear equations. Then, in some examples the mentioned wavelets are compared with each other.

Keywords: First kind Volterra and Fredholm integral equation; Galerkin method; Collocation method; Haar, Legendre, CAS, CFK, CSK wavelets.

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1. Introduction.

The theory and application of integral equations are the important subjects in applied sciences. Integral equations are used as mathematical models for different physical situations. These integral equations also occur as reformulations of other mathematical problems such as ordinary and partial differential equations. For this reason, it is important to explain the appropriate methods with high accuracy to solve these kinds of integral equations numerically.

Many inverse problems in science and engineering lead to the solution of the following integral equations of the first kind[2,9],

$$\int_a^b k(x,t)f(t)dt = g(x)\,,\ -\infty < a \le x \le b < \infty, \quad (1)$$
$$\int_a^s k(s,t)f(t)\mathrm{d}t = g(s)\,,\ -\infty < a \le s \le b < \infty, \quad (2)$$

where, $g(s)$ and $k(s,t)$ are known functions and $f(t)$ is the unknown function to be determined. In general, these kinds of integral equations are ill-posed for a given kernel k and driving term g. For this reason, the special methods should be introduced to solve them. In recent years, several numerical methods for approximating the solution of the first kind of integral equation are known. Among these methods, the methods based on the wavelets are more attractive and considerable. The wavelets technique allows the creation of very fast algorithms when compared to algorithms which are ordinarily used. Various wavelet basis are applied. In [8] Maleknejad used Legendre wavelets, Xufeng Shang in [11] applied the Legendre multi wavelets. In [6] and [5] Lepik and Gu proposed non-uniform Haar wavelets and Trigonometric Hermit wavelets too. Recently, wavelets basis are applied in order to solve various kinds of integral equations [1,2,8]. In [7] the first kind integral equations of Volterra type are solved by using Haar wavelet. In this paper, we propose the method which was presented in [3 , 4] by applying other kinds of well known wavelets such as Legendre and Chebyshev wavelets and compare the results with Haar wavelet.
In this paper, we present the application of Legendre [8], CFK [1], CSK and CAS [12] wavelets as a basis functions in Galerkin method for numerical solution of the Eq. (1) and compare them with each other. The method is tested by using of some numerical examples. Also, we apply and compare the Legendre, CFK and CAS wavelets as the basis functions in collocation method in order to solve the Eq. (2).
At first, in section 2, we introduce the wavelets which are used as basis functions to approximate the unknown function. Then, in sections 3 and 4 we remind the Galerkin and the collocation methods for computing the unknown function using the mentioned wavelets. Finally, in section 5, we solve some Fredholm and Volterra integral equations of the first kind with different wavelets and compare them with each other.

2. Wavelets and their properties

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [8],

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a,b\epsilon\mathbb{R}\ \ ,a \neq 0$$

where, $\psi_{a,b}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$. We consider the parameters a and b as discrete values $a = a_0^{-k}\,, b = nb_0a_0^{-k}\,, a_0 > 1\,, b_0 > 0$ where, n and k are positive integers. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}(t)$ form an orthonormal basis [8]. In this work, we use the following wavelets. In the sequel, the notation $(.,.)$ means the inner product and $(.,.)_w$ denotes the inner product with respect to the weight function $w(t)$.

2.1. Legendre wavelets

Consider the well-known Legendre polynomials of order m, $L_m(t)$, which are orthogonal with respect to the weight function $w(t) = 1$ and derived from the following recursive formula:

$L_0(t) = 1\,, L_1(t) = t,$

$L_{m+1}(t) = \frac{2m+1}{m+1}tL_m(t) - \frac{m}{m+1}L_{m-1}(t), \quad m = 1,2,3,\ldots$

Legendre wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments; $k = 2,3,\ldots$, $n = 1,2,3,\ldots,2^{k-1}$, $m = 0,1,2,\ldots,M-1$. m is the order of Legendre polynomials and M is a fixed positive integer. They are defined on the interval $[0,1)$ as follows:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{L}_m\left(2^k t - 2n + 1\right) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & otherwise, \end{cases} \quad (3)$$

where, $\tilde{L}_m(t) = (2m+1)^{\frac{1}{2}} L_m(t)$.

2.1. CFK wavelets

CFK wavelets $\psi_{n,m} = \psi(k,n,m,t)$ have four arguments, $n = 1,2,\ldots,2^{k-1}$, k can assume any positive integer, m is the degree of Chebyshev polynomials of the first kind and t denotes the time.

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m\left(2^k t - 2n + 1\right) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0 & otherwise, \end{cases}$$

where,

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \sqrt{\frac{2}{\pi}}\, T_m(t) & m > 0, \end{cases} \quad (4)$$

and $m = 0,1,\ldots,M-1$, $n = 1,2,\ldots,2^{k-1}$.

In Eq.(4), $T_m(t)$ are Chebyshev polynomials of degree m which are orthogonal with respect to the function $w(t) = 1/\sqrt{1-t^2}$,on the interval $[-1,1]$,and satisfy the following recursive formula:

$T_0(t) = 1, T_1(t) = t,$

$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t)$, $m = 1,2,\ldots$

2.2. CSK wavelets

CSK wavelets $\psi_{n,m} = \psi(k,n,m,t)$ also have four arguments, $n = 1,2,\ldots,2^{k-1}$, k can assume any positive integer, m is the degree of Chebyshev polynomials of the second kind [10] and t denotes the time.

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{U}_m\left(2^k t - 2n + 1\right) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0 & otherwise, \end{cases}$$

where,

$$\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}}\, U_m(t)\, , m \ge 0 \quad (5)$$

and $m = 0,1,\ldots,M-1$, $n = 1,2,\ldots,2^{k-1}$.

In Eq.(5), $U_m(t)$ are Chebyshev polynomials of degree m which are orthogonal with respect to the function $w(t) = \sqrt{1-t^2}$,on the interval $[-1,1]$,and satisfy the following recursive formula:

$U_0(t) = 1, U_1(t) = 2t,$

$U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$, $m = 1,2,\ldots$

The set of $\{\psi_{n,m}(t)\}$ are composed and be expanded by Legendre, FKC or SKC wavelet series as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)\, , \quad (6)$$

where, $c_{n,m} = \left(f(t), \psi_{n,m}(t)\right)_w$.

If the infinite series in (6) is truncated, then (6) can be written as

$$f(t) \sim \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t).$$

2.3. CAS wavelets

CAS wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments, $n = 0,1,\ldots,2^k - 1$, k can assume any nonnegative integer, m is any integer and t is the normalized time and weight function is $w(t) = 1$. They are defined on the interval $[0,1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} CAS_m(2^k t - n) & \frac{n}{2^k} \le t < \frac{n+1}{2^k} \\ 0 & otherwise, \end{cases} \quad (7)$$

where,

$$CAS_m(t) = \cos(2m\pi t) + \sin(2m\pi t). \quad (8)$$

The set of $\{\psi_{n,m}(t)\}$ are composed and be expanded by CAS wavelet series as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} c_{n,m} \psi_{n,m}(t), \quad (9)$$

where $c_{n,m} = \left(f(t), \psi_{n,m}(t)\right)$.

If the infinite series in (9) is truncated, then (9) can be written as

$$f(t) \sim \sum_{n=1}^{2^{k-1}} \sum_{m=-M}^{M} c_{n,m} \psi_{n,m}(t).[12] \quad (10)$$

3. Galerkin Method

In Galerkin method, we choose a sequence of finite dimensional subspaces $X_n \subset X = L^2[0,1]$ with $\dim(X_n) = d_n$. Suppose that $X_n = span\{\varphi_1, \varphi_2, \ldots, \varphi_{d_n}\}$ where $\{\varphi_i\}_{i=1}^{d_n}$ are the orthonormal Legendre or Chebyshev wavelet basis and y_n is a function belongs to X_n as an approximate solution of Eq. (1). Hence, we can write,

$$y_n(t) = \sum_{j=1}^{d_n} c_j \varphi_j(t).$$

By substituting into Eq. (1), we have

$$r_n(x) = \int_a^b k(x,t) y_n(t) dt - g(x) = \int_a^b k(x,t) \sum_{j=1}^{d_n} c_j \varphi_j(t) dt - g(x),$$
$$a \le x \le b,$$

where, r_n is called the residual in the approximation of Eq.(1) [2].

Now, to determine unknown coefficients, we impose the following requirements:

$$\left(r_n(x), \varphi_i(x)\right)_w = 0 \quad i = 1,2,\ldots,d_n, n \ge 1, a \le x \le b.$$

Thus, we obtain the system of algebraic equation:

$$AC = G. \quad (11)$$

In this system, the unknowns vector is $C = [c_1, c_2, \ldots, c_{d_n}]^T$ and the elements of A and G are:

$$a_{ij} = \left(\int_a^b k(x,t)\varphi_i(t) \mathrm{d}t, \varphi_j(x)\right)_w, \quad g_i = \left(g(x), \varphi_i(x)\right)_w, \quad i,j = 1,2,\ldots,d_n.$$

In this section, we apply the Galerkin method based on the mentioned wavelets as basis for solving Eq.(1). In the tables, the absolute errors are computed for each one of the wavelets. [a,b]=[0,1]

Algorithm3. 1.

Step 1: Consider the basic functions of Legendre, CAS, FKC or SKC wavelets $\{\varphi_i(x)\}_{i=1}^{n}$

Step 2: Calculate matrix $A, n \times n$, with entry $a_{ij} = \int_0^1 \int_0^1 w(x) k(x,t) \varphi_i(t) \varphi_j(x) dt dx$ and matrix $F, n \times 1$ with entry $f_i = \int_0^1 w(x) \varphi_i(x) g(x) dx$ and then solve the system of $AC = F$ to obtain C.

Step 3: Approximate $y(x)$ in Eq.(1) with $y_n(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, where c_i is i-th entry of C.

Step 4: Compute the error of the method using $E = |f(x) - f_n(x)| = |f - \tilde{f}|$ where f and $\tilde{f}$ denote the exact solution and the numerical solution, respectively.

4. Collocation Method

In this section, as we mentioned in the previous section to approximate the unknown function in the integral Eq. (2). Let $f_n(t) = \sum_{j=1}^{d_n} c_j \varphi_j(t)$ be the approximate solution of Eq. (2). Now, by substituting in integral Eq. (2) we get the following residual function:

$$r_n(s) = \int_a^s k(s,t) \sum_{j=1}^{d_n} c_j \varphi_j(t)\, \mathrm{d}t - g(s).$$

Now, for determining unknown coefficients c_j we can choose an expansion method [6]. We choose the collocation method to find c_j. In the collocation method, we select some collocation points then put the residual equation in these points equal to zero. So, we have

$$r_n(s_i) = \int_a^{s_i} k(s_i,t) \sum_{j=1}^{d_n} c_j \varphi_j(t)\, \mathrm{d}t - g(s_i) = 0,$$

where, the collocation points are:

$$s_i = a + \frac{i(b-a)}{M2^{k-1}}, i = 1,2,\dots,d_n.$$

Therefore, the following system of linear equations is formed [6,1],

$$AX = G, \tag{12}$$

where, the elements of A and G are:

$$a_{ij} = \left[\int_a^{s_i} k(s_i,t)\varphi_j(t)dt\right]_{i,j=1}^{d_n} \text{ and } b_i = [g(s_i)]_{i=1}^{d_n}.$$

In the Legendre, CFK or CSK wavelet $d_n = M2^{k-1}$.

Algorithm 4.1.

Step 1: Consider the basic functions of Haar [5], Legendre [8] and Chebyshev [1] wavelets as $\{\varphi_i(x)\}_{i=1}^{d_n}$.

Step 2: Calculate matrix A with entry $a_{ij} = \int_0^{s_i} K(s_i,t)\, \varphi_j(t)dt$ and vector G with entry $G_i = g(s_i)$ and then solve the system $AC = G$ to obtain C.

Step 3: Approximate $f(x)$ in Eq.(1) with $f_n(t) = \sum_{i=1}^{d_n} c_i \varphi_i(t)$, where c_i is i-th entry of C.

Step 4: Compute the error of the method by using $E = |f(x) - f_n(x)| = |f - \tilde{f}|$, where f and $\tilde{f}$ denote the exact solution and the numerical solution respectively.

5. Illustrative examples

In this section, we apply the collocation method based on the mentioned wavelets as basis for solving Eq. (1). In the tables, the absolute errors are computed for each one of the wavelets. The programs have been provided with Mathematica 6 according to the algorithms 1 and 2 in the interval $[a,b] = [0,1]$. We use the points mentioned in (6) and construct the system (7) in the algorithm.

Example 5.1. Consider the following Fredholm integral equation of the first kind:

$$\int_0^1 k(x,t)y(t)dt = e^x + (1-e)x - 1,$$

$$k(x,t) = \begin{cases} t(x-1) & x > t, \\ x(t-1) & x \le t. \end{cases}$$

The exact solution is $y(x) = e^x$. Table 1 illustrates the results of the example.

Table 1. Comparison of Legendre,FKC, CKC and CAS wavelets methods with $\mathrm{k} = 1$ *and* $\mathrm{M} = 2$

x	$\lvert f - \tilde{f}\rvert$			
	CFK wavelet	CSK wavelet	CAS wavelet	Legendre wavelet
0	0.163509	0.137302	0.287703	0.468045
0.1	0.097782	0.0771119	0.0557322	0.288083

0.2	0.0431175	0.0279823	0.0378095	0.119181
0.3	0.0006763	0.00892308	0.0380718	0.037496
0.4	0.028255	0.0323186	0.057077	0.180663
0.5	0.0422557	0.0407835	0.492577	0.3089
0.6	0.0397554	0.0327474	0.10781	0.135503
0.7	0.0190187	0.00647491	0.0721787	0.0561311
0.8	0.0218722	0.0399519	0.0726111	0.267919
0.9	0.0850372	0.10865262	0.110028	0.501982

Table 1 shows that the results of the example for FKC, SKC and CAS wavelets are better than the result of Legendre wavelets.

Example 5.2. Consider the following Fredholm integral equation:

$$\int_0^1 \sqrt{(x^2+t^2)}\, y(t)dt = \frac{\sqrt[3]{(1+x^2)} - x^3}{3},$$

with exact solution $y(x) = x$. The numerical results are represented in tables 2 and 3.
With comparing the results of tables 2 and 3, we can observe that the growth of the error in FKC and SKC wavelets, as k and M increase, is less than the growth in the Legendre wavelets.

Table 2. Comparison of Legendre, FKC and SKC wavelets method with $k = 2$ and $M = 2$

x	$\lvert f-\tilde{f}\rvert$		
	CFK wavelet	CSK wavelet	Legendre wavelet
0	4.79773e − 9	2.11279e − 9	4.85723e − 15
0.1	1.80309e − 9	6.65198e − 10	7.21645e − 16
0.2	1.19155e − 9	7.82395e − 10	6.30052e − 15
0.3	4.1862e − 9	2.22999e − 9	1.18794e − 14
0.4	7.18084e − 9	3.67758e − 9	1.7486e − 14
0.5	1.21431e − 8	7.22839e − 9	4.13003e − 14
0.6	8.37393e − 9	4.95011e − 9	2.80886e − 14
0.7	4.60473e − 9	2.67183e − 9	1.4988e − 14
0.8	8.3553e − 10	3.93559e − 10	1.88738e − 15
0.9	2.93367e − 9	1.88472e − 9	1.12133e − 14

Table 3. Comparison of Legendre, CFK and CSK wavelets method with $k = 3$ and $M = 3$

x	$\lvert f-\tilde{f}\rvert$		
	CFK wavelet	CSK wavelet	Legendre wavelet
0	1.50084e − 6	3.61294e − 6	1.09971e − 8
0.1	5.70364e − 5	8.09372e − 7	6.94946e − 8
0.2	0.000017645	0.0000382199	3.793e − 7
0.3	0.00053280	0.000847741	0.000051191
0.4	0.0008800	0.00125236	0.000112012
0.5	0.0397811	0.0568369	0.0049414
0.6	0.0153297	0.0207851	0.00206823
0.7	0.0188161	0.0238855	0.00280601
0.8	0.0184679	0.0233272	0.00276108
0.9	0.0071588	0.00857818	0.00113444

Example 5.3. In this example, we consider the following Volterra integral equation:

$$\int_0^s \frac{1}{s^2+t^2} f(t)\mathrm{d}t = \frac{1}{4}\left(\pi\left(\frac{1}{s}-s\right)+4s-\ln(4)\right).$$

The exact solution is $f(t) = t^2 - t + 1$. Table 4 illustrates the results of this example.

Table 4. Comparison of Haar , Legendre and CFK wavelets with $\mathrm{k}=3, \mathrm{M}=4$

x	$\lvert f-\tilde{f}\rvert$		
	Haar wavelet	Legendre wavelet	CFK wavelet
1/16	0.027424	1.11022e − 16	0
2/16	0.0223668	2.22045e − 16	2.22045e − 16
3/16	0.0184296	3.33067e − 16	0
4/16	0.0145393	1.88738e − 15	2.9976e − 15
5/16	0.0106433	9.99201e − 16	1.33227e − 15
6/16	0.00674223	7.77156e − 16	2.22045e − 15
7/16	0.00283853	1.55431e − 15	1.9984e − 15
8/16	0.00106646	8.54872e − 15	9.21485e − 15
9/16	0.00497209	1.55431e − 15	2.9976e − 15
10/16	0.0087803	2.22045e − 16	5.55112e − 15
11/16	0.0127841	1.66533e − 15	5.88418e − 15
12/16	0.0166903	1.60982e − 14	5.10703e − 14
13/16	0.0205966	5.88418e − 15	3.88578e − 15
14/16	0.0245028	1.66533e − 14	1.18794e − 14
15/16	0.0284091	1.25455e − 14	8.32667e − 15

Example 5.4. In this example, we consider the following Volterra integral equation:

$$\int_0^s \frac{e^{s-t}}{1+s^2} f(t)\mathrm{d}t = \frac{4\pi\cos(4\pi s)+\sin(4\pi s)-4\pi e^s}{(1+s^2)(1+16\pi^2)}.$$

The exact solution is $f(t) = \sin(4\pi t)$. Table 5 shows the result for example 5.4.

Table 5. Comparison of Haar, Legendre and CFK wavelets with $\mathrm{k}=3, \mathrm{M}=4$

x	$\lvert f-\tilde{f}\rvert$		
	Haar wavelet	Legendre wavelet	CFK wavelet
1/16	0.154976	0.00962564	0.00962564
2/16	0.210496	0.085175	0.085175
3/16	0.495292	0.130583	0.130583
4/16	0.483232	0.0940333	0.0940333
5/16	0.160889	0.0650563	0.0650563
6/16	0.238234	0.116735	0.116735
7/16	0.388895	0.126098	0.126098
8/16	0.123382845	0.10521	0.10521
9/16	0.41821326	0.432794	0.432794
10/16	0.86407874	0.749734	0.749734
11/16	0.87067629	0.712492	0.712492
12/16	0.38189203	0.324229983041	0.32422998304
13/16	0.304440788	0.290656637053	0.2906566370533

14/16	*0.720956486*	*0.635726132474*	*0.6357261324742*
15/16	*1.184604694*	*0.539405627967*	*0.5394056279676*

As we observe in the tables 4 and 5, the result of the examples for Legendre and Chebyshev wavelets is more accurate than the result of Haar wavelet.

6. Conclusion

The aim of this work is to propose a method for solving Volterra integral equation of the first kind using Legendre and Chebyshev wavelets. The presented method converts the integral equation into a system of linear algebraic equations. We applied the Legendre and Chebyshev wavelets as basis function and observed that these wavelets can behave better than the Haar wavelet on $[0,1]$.

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References.

[1] E. Babolian, F. Fattahzadeh, Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Compt., (2007), pp. 1016–1022.

[2] L. M. Delves, J. L. Mohamed, Computational Methods for integral equations, Cambridge University Press, (1985).

[3] M. A. Fariborzi Araghi, M. Bahmanpour, Numerical Solution of Fredholm Integral Equation of the First kind using Legendre, Chebyshev and CAS wavelets, International J. of Math. Sci. & Engg. Appls. (IJMSEA) Vol. 2 No. IV (2008), pp. 1–9.

[4] M. A. Fariborzi Araghi, M. Bahmanpour, Numerical Solution of Fredholm Integral Equation of the First Kind by Using Chebyshev Wavelets, Proceeding of the International Conference of Approximation on Scientific Computing (ICASC'08), 26–30 Oct. (2008), ISCAS, Beijing, China.

[5] J. S. Gu, W. S. Jiang, The Haar wavelets operational matrix of integration, Int. J. Sys. Sci. 27, (1996), pp. 623–628.

[6] U. Lepik, Solving integral and differential equations by the aid of non-uniform Haar wavelets, Appl. Math. Compt. 198 (2008), pp. 326–332.

[7] K. Maleknejad., R. Mollapourasl and M. Alizadeh, Numerical solution of Volterra type integral equation of the first kind with wavelet basis, Appl. Math. Compt. 194 (2007), pp. 400–405.

[8] M. Maleknejad, S. Sohrabi, Numerical solution of Fredholm integral equation of the first kind by using Legendre wavelets, Appl. Math. Compt., (2007), pp. 836–843.

[9] M. Rabbani, R. Jamali, Solving Nonlinear System of Mixed Volterra-Fredholm Integral Equations by Using Variational Iteration Method, The Journal of Mathematics and Computer Science, Vol .5 No.4 (2012), pp. 280–287.

[10] T. J. Rivlin, Chebyshev Polynomials, John Wiley and Sons, Second Edition, (1990).

[11] X. Shang and D. Han, Numerical solution of Fredholm integral equation of the first kind by using linear Legendre multi-wavelets, Appl. Math. Compt. 191 (2007), pp. 440–444.

[12] S. Yousefi, A. Banifatemi., Numerical solution of Fredholm integral equations by using CAS wavelets, Appl. Math. Compt.,(2006), pp. 458–463.