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COMMON FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS IN COMPLEX-VALUED METRIC SPACES

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Abstract

The purpose of this paper is to prove common fixed point theorems for a pair of mappings satisfying a quasi-contraction condition in a complex-valued metric space (X, d). For this, we have defined the '*max*' function for the partial order $\leq$ in complex-valued metric d.

Keywords: Common fixed point, contraction mapping, contractive condition, Banach contraction condition, Complex-valued metric space.

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1. Introduction.

An ordinary metric d is a real-valued function from a set X×X into R, where X is a nonempty set. That is, d:X×X→R. A complex number $z \varepsilon C$ is an ordered pair of real numbers, whose first co-ordinate is called *Re(z)* and second coordinate is called *Im(z)*. Thus a complex-valued metric *d* would be a function from a set $X\times X$ into C, where X is a nonempty set and C is the set of complex number. That is, $d:X\times X\rightarrow R$. Define a partial order ≤ on C as follows; let $z_1, z_2 \varepsilon C$.

$$z_1 \leq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2),$

(ii) $Re(z_1) < Re(z_2),\ Im(z_1 = Im(z_2),$

(iii) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2),$

(iv) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2).$

In (i), (ii) and (iii), we have $|z_1|<|z_2|$. In (iv), we have $|z_1|=|z_2|$. So that, $|z_1|\leq|z_2|$. In particular, z_1 *not*≤ z_2 if $z_1 \neq z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1|<|z_2|$. Also $z_1<z_2$ if only (iii) satisfy. Further,

$$0 \leq z_1 \ \ not\leq\ z_2 \text{ implies } |z_1|<|z_2|,$$

$$z_1 \leq z_2, \quad z_2 < z_3 \text{ implies } z_1 < z_3.$$

From this definition of complex-valued metric *d*, Azam *et. al*. [1] defined the complex-valued metric space (*X, d*) in the following way:

Definition 1.1. *Let X be a nonempty set. Suppose that the mapping d:X×X→C satisfies the following conditions:*

(C1) $0 \leq d(x, y)$ *for all* $x, y \varepsilon X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(C2) $d(x, y) = d(y, x)$ *for all* $x, y \varepsilon X$;

(C3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \varepsilon X$.

Then d is called a complex-valued metric in X, and (X, d) is called a complex-valued metric space.

A point $x \varepsilon X$ is called an *interior point* of A subseteq X if there exists $r \varepsilon C$, where $0 < r$, such that

$$B(x, r) = \{y \varepsilon X: d(x, y) < r\} \text{ subseteq } A.$$

A point $x \varepsilon X$ is called a *limit point* of A subseteq X, if for every $0 < r \varepsilon C$,

$$B(x, r) \cap (AX) \neq \varphi.$$

The set A is called *open* whenever each element of A is an interior point of A. A subset B is called closed whenever each limit point of B belongs to B.

The family $F := \{B(x, r): x \varepsilon X, 0 < r\}$ is a sub-basis for a Hausdorff topology τ on X. Let $\{x_n\}$ be a sequence in X and $x \varepsilon X$. If for every $c \varepsilon C$ with $0 < c$, there exists $n_0 \varepsilon N$ such that for all $n > n_0$, $d(x_n, x) < c$, then $\{x_n\}$ is called *convergent*. Also, sequence $\{x_n\}$ converges to x (written as, $x_n \rightarrow x$ or

$lim_{n\rightarrow\infty} x_n = x$); and x is the *limit point* of $\{x_n\}$. The sequence $\{x_n\}$ *converges* to x if and only if

$lim_{n\rightarrow\infty}|d(x_n, x)| = 0$. If for every $c \varepsilon C$ with $0 < c$, there exists $n_0 \varepsilon N$ such that for all $n > n_0$,

$d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is called *Cauchy sequence* in (X, d). If every Cauchy sequence converges in X, then X is called a *complete complex-valued metric space*. The sequence $\{x_n\}$ is called *Cauchy* if and only if $lim_{n\rightarrow\infty}|d(x_n, x_{n+m})| = 0$.

Definition 1.2. *We define the 'max' function for the partial order relation* $\leq$ *by:*

(1) $max\{z_1, z_2\} = z_2$ *if and only if* $z_1 \leq z_2$,

(2) $z_1 \leq max\{z_2, z_3\}$ *implies* $z_1 \leq z_2$, *or* $z_1 \leq z_3$.

Using Definition 1.2 we have the following Lemma:

Lemma 1.3. Let $z_1, z_2, z_3, \ldots\ \varepsilon$ C and the partial order relation $\leq$ is defined on C. Then

(i) If $z_1 \leq max\{z_2, z_3\}$ then $z_1 \leq z_2$ if $z_3 \leq z_2$;

(ii) If $z_1 \leq max\{z_2, z_3, z_4\}$ then $z_1 \leq z_2$ if $max\{z_3, z_4\} \leq z_2$;

(iii) If $z_1 \leq max\{z_2, z_3, z_4, z_5\}$ then $z_1 \leq z_2$ if $max\{z_3, z_4, z_5\} \leq z_2$, and so on.

Since (X, d) is a complex-valued metric space, the 'usual metric' in R is not definable; as shown in Example 7 [1]. Keeping this in view, we need to generalize the Banach contraction principal [2] in complex-valued metric space, as follows:

Theorem 1.4. *Let (X, d) be a complete, complex-valued metric space and T be a mapping of X into itself, satisfying:*

$$d(Tx, Ty) \leq k\, d(x, y), \ \text{for all } x, y\ \varepsilon\ X; \tag{1.1}$$

where k is a constant in (0,1). Then T has a unique common fixed point in X.

Proof. For an arbitrary x_0 in X, we have $T^n x_0 = x_n$. The sequence $\{x_n\}$ is Cauchy. For, we have since

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq k\, d(x_0, x_1),$$

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leq k\, d(x_1, x_2) \leq k^2\, d(x_0, x_1),$$

....

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kd(x_{n-1}, x_n) \leq k^n\, d(x_0, x_1) \tag{A}$$

Hence for any $m > n, \quad m, n\ \varepsilon\ N$

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots + d(x_{n+m-1}, x_{n+m})$$

$$\leq k^n\, d(x_0, x_1) + k^{n+1}\, d(x_0, x_1) + \ldots\ldots + k^{n+m-1}\, d(x_0, x_1)$$

$$\leq k^n.d(x_0, x_1)/(1-k) \leq d(x_0, x_1) , \quad \text{as } 0< k<1.$$

Therefore $|d(x_n, x_{n+m})| \leq \{k^n/(1-k)\}.|d(x_0, x_1)| \rightarrow 0$; as $m, n \rightarrow\infty$. Thus $\{x_n\}$ is a Cauchy sequence. The completeness of X implies that sequence $\{x_n\}$ converges to some $x \varepsilon X$. We claim that $x = Tx$, otherwise $|d(x, Tx)| = |z| > 0$, and we would then have

$$|d(x, Tx)| = |z| \leq |d(x, x_n) + d(x_n, Tx)| = |d(x, x_n) + d(Tx_{n-1}, Tx)|$$

$$\leq |d(x, x_n)| + |d(Tx_{n-1}, Tx)| = |d(x, x_n)|+k|d(x_{n-1}, x)| \rightarrow 0 \text{ as } n\rightarrow\infty.$$

Thus $x = Tx$. The uniqueness of x follows easily. For, if x' be another fixed point then

$$d(x, x') \leq d(x, Tx) + d(Tx, x') = d(Tx, Tx') \leq k.d(x, x'), \quad \text{by (1.1).}$$

Taking modulus in above, we have

$$|d(x, x')| \leq k|d(x, x')| < |d(x, x')|,$$

a contradiction. Thus x is unique fixed point in X. This completes the proof. □

2. Main Results

Theorem 2.1. *Let (X, d) be a complete complex-valued metric space and mappings $S, T: X\rightarrow X$ satisfying:*

$$d(Sx, Ty) \leq h \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\} \tag{2.1}$$

for all x, y ε X; where 0 < h < ½. Then S and T have a unique common fixed point in X.

Proof. Choose an arbitrary point x_0 in X. Sequence $\{x_n\}$ can be formed in X such that $Sx_0 = x_1$, $Tx_1 = x_2$, $Sx_2 = x_3$, $Tx_3 = x_4$, ...

$$Sx_{2n} = x_{2n+1},\ Tx_{2n+1} = x_{2n+2}. \tag{2.2}$$

We show that the sequence $\{x_n\}$ is Cauchy. For, putting $x = x_{2k}$ and $y = x_{2k+1}$ in (2.1), we have

$$d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$$

$$\leq h\max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, Sx_{2k}), d(x_{2k+1}, Tx_{2k+1}), d(x_{2k}, Tx_{2k+1}), d(x_{2k+1}, Sx_{2k})\}$$

$$= h\max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), d(x_{2k}, x_{2k+2}), 0\}, \text{ by (2.2)} \tag{B}$$

$$\leq h\max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), d(x_{2k}, x_{2k+1})+d(x_{2k+1}, x_{2k+2}), 0\}$$

whence,

$$d(x_{2k+1}, x_{2k+2}) \leq h\,[d(x_{2k}, x_{2k+1})+d(x_{2k+1}, x_{2k+2})], \text{ as other co-ordinates are less}$$

i.e., $d(x_{2k+1}, x_{2k+2}) \leq [h/(1-h)].d(x_{2k}, x_{2k+1})$.

Similarly, by putting $x = x_{2k+2}$ and $y = x_{2k+1}$ in (2.1), we have

$$d(x_{2k+2}, x_{2k+3}) \leq [h/(1-h)].d(x_{2k+1}, x_{2k+2}).$$

Hence for each $n = 1, 2, 3,$ .... we have

$$d(x_n, x_{n+1}) \leq H.d(x_{n-1}, x_n), \tag{C}$$

where $0 < H = h/(1-h) < 1$. From this we have, inductively

$$d(x_n, x_{n+1}) \leq H.d(x_{n-1}, x_n) \leq H^2.d(x_{n-2}, x_{n-1}) \leq ...... \leq H^n.d(x_0, x_1) \quad (2.3)$$

Thus for any $m > n$, $m, n \varepsilon N$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{m-1}, x_m),$$

$$\leq [H^n + H^{n+1} + H^{n+2} + ... + H^{m-1}].d(x_0, x_1), \quad \text{by (2.3)}$$

$$\leq [H^n/(1\text{-}H)].d(x_0, x_1),$$

So that $|d(x_n, x_m)| \leq |\{H^n/(1\text{-}H)\}.d(x_0, x_1)| \rightarrow 0$ as $n \rightarrow \infty$.

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, therefore $\{x_n\}$ converges to some point u (say) in X. We claim that u is a fixed point of S. Otherwise $u \neq Su$ and $|d(u, Su)| = |z| > 0$. From triangle inequality and using (2.1), we have successively

$$d(u, Su) \leq d(u, x_{2k+2}) + d(x_{2k+2}, Su)$$

$$\leq d(u, x_{2k+2}) + d(Tx_{2k+1}, Su)$$

$$\leq d(u, x_{2k+2}) + h \max\{d(u, x_{2k+1}), d(u, Su), d(x_{2k+1}, Tx_{2k+1}), d(u, Tx_{2k+1}), d(x_{2k+1}, Su)\}.$$

Taking magnitude in above, and using $|a+b| \leq |a|+|b|$, for all a, b ε C, we have

$$|d(u, Su)| \le d(u, x_{2k+2}) + h\, max\{|d(u, x_{2k+1})|, |d(u, Su)|, |d(x_{2k+1}, Tx_{2k+1})|, |d(u, Tx_{2k+1})|, |d(x_{2k+1}, Su)\}|.$$

Letting $n \to\infty$ we have

$$|z| = |d(u, Su)| \le 0 + h\, max\{0, |z|, 0, 0, |z|\} = h.|z| < |z|,$$

a contradiction. Thus $|z| = |d(u, Su)| = 0$, yielding $u = Su$.

Further, since X is complete, there exist some v in X such that $v = Tu$. We claim that $u = v$. If not, then from (2.1), we have

$$d(u, v) = d(Su, Tu) \le h\, max\{d(u, u), d(u, Su), d(u, Tu), d(u, Tu), d(u, Su)\}$$

$$\le h\, max\{0, 0, d(u, v), d(u, v), 0\} = h\, d(u, v).$$

Whence, on taking magnitude, $|d(u, v)| \le |h.d(u, v)| < |d(u, v)|$, a contradiction.

Thus $u = v = Tu = Su$, and u is the common fixed point of S and T. For uniqueness of common fixed point, let u_0 be another common fixed point of S and T. Then from (2.1), we have

$$d(u, u_0) = d(Su, Tu_0) \le h\, max\{d(u, u_0), d(u, Su), d(u_0, Tu_0), d(u, Tu_0), d(u_0, Su)\},$$

whence,

$$|d(u, u_0)| \le h\, max\{|d(u, u_0)|, 0, 0, |d(u, u_0)|, |d(u_0, u)|\} = h\, |d(u, u_0)| < |d(u, u_0)|,$$

a contradiction. Thus S and T have unique common fixed point. This completes the proof. □

If the function '*max*' has only three variables, as shown in (2.4) below, then we have the following theorem:

Corollary 2.2. Let (X, d) be a complete complex-valued metric space and mappings $S, T{:}X \to X$ satisfying:

$$d(Sx, Ty) \leq h \max\{d(x, y), d(x, Sx), d(y, Ty)\} \tag{2.4}$$

for all $x, y \in X$; where $0 < h < 1$. Then S and T have a unique common fixed point in X.

Proof. In this case, eq.(B) reduces to:

$$d(x_{2k+1}, x_{2k+2}) \leq h \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k}, x_{2k+1})$$

so, eq.(C) reduces to:

$$d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n), \text{ where } 0 < h < 1.$$

This is eq.(A). Further proof runs smoothly as Theorem 1.4 and Theorem 2.1. □

Remark. By putting S = T in above corollary, we obtain Theorem 1.4. Thus, Corollary 2.2 is a generalization of Theorem 1.4.

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