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# FEW COMMON FIXED POINT RESULTS FOR WEAKLY COMMUTING MAPPINGS

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### Abstract

In this paper we give a common fixed point  result  for a mapping of  f-contractive mappings, which generalize the results of Chang [2], Hadzic [4] and Sessa et.al [10]. We also extend the results of Fisher [3], Imdad  and  Khan [7], Iseki [6], Singh [11], Singh and Tiwari [12].

**Key words** :  Weakly commuting maps, f-contraction, common fixed point.

## 1.  Introduction

Let  T  be a mapping of a nonempty set X into itself . Then T is said to have a fixed point if there exists a point u in X such that Tu = u.

Two self mappings A and S of a metric space (X, d) are called weakly commuting if
$$d(ASx, SAx) \leq d(Ax, Sx) \quad \text{for all } x \in X.$$

Let (X, d)  be a complete metric space, N  the set of all positive integer and let F be the set of all
real functions  f : $[0, \infty) \rightarrow [0, \infty)$  such that every  f $\in$ F is non decreasing, upper semi continuous and f(t) < t for  any t > 0.
For a family of mappings Hadzic [4]   proved the following result.

**Theorem 1.1** Hadzic [4] Let S, T: X → X be two continuous mappings and $\{A_i\}_{i \in N}$ be a family of self mappings of X satisfying

(i)      $A_i(X) \subseteq S(X) \cap T(X)$   for each  $i \in N$.

(ii)      $A_i$ commutes with S and T for each  $i \in N$.

(iii)      $d(A_i x, A_j y) \le q \, d(Sx, Ty)$   for any  $x,y \in X$  and  $i, j \in N$, $i \ne j$, where  $0 \le q < 1$. Then $\{A_i\}_{i \in N}$,

S and T  have a unique common fixed  point.

Generalizing  the results of  Husain  and  Sehgal [5]  and  Iseki [6], Chang [2] established  the following result  for a family of f-contraction  mappings :

**Theorem 1.2** [2]   Let  S, T: X → X  be two continuous maps. Then S  and T  have a common fixed point w if and only if  there exists  two self mappings A, B of X and a function $f \in F$ such  that

(iv)    $A(X) \cup B(X) \subseteq S(X) \cap T(X)$ ,

(v)    both A and B  commute with S and T ,

(vi)    $d(Ax, By) \le f \, (\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), [d(Sx, By) + d(Ty, Ax)]/2\})$ ,

for any $x,y \in X$. Further w is the unique common fixed point of  A, B, S and  T.

Unifying the above results of  Hadzic [4] and Chang [2], Sessa et.al. [10] proved two common fixed point results for weakly commuting mappings. We further generalize the condition satisfied by the mappings and obtain some new results. Before giving our main result we prove few lemmas.

**Lemma 1.1**  (Chang [2])  Let  $f \in F$  and  $t_o > 0$. If  $t_{n+1} \le f(t_n)$  for  $n \in N$, then  the sequence $\{t_n\}_{n \in N}$ converges  to  zero.

**Proof.**  See Chang [1].

Let  S,T: (X,d) → (X,d) and  $\{A_i\}_{i \in N}$ be a  family of  self mappings  of  X  satisfying  condition (i) and the following generalized inequality  (4)

(4)    $ad \, (A_i x, A_j y) + bd(Ty, A_j y) - \min\{d(Ty, A_i x), d(Sx, A_i x)\}$

$\qquad \le f(\max \{d(Sx, Ty), d(Sx, A_i x), d(Ty, A_j y), d(Ty, A_i x), [d(Sx, A_j y) + d(Ty, A_i x)] /2\})$

for all  $x,y \in X$, $i,j \in N$, $i \ne j$, where  $f \in F$ and $a+b \ge 1$. Then for any arbitrary point  $x_o \in X$, we  get a point  $x_1 \in X$, guaranteed  by (i), such  that  $Tx_1 = A_1 x_o$  and  for this $x_1$, there exists a point $x_2 \in X$ such that  $Sx_2 = A_2 x_1$ and so on. Inductively we get a sequence $\{y_n\}_{n \in N}$  such  that

(5)                $y_{2n+1} = Tx_{2n+1} = A_{2n+1} x_{2n}$  for  every $n \in N_o = N \cup \{0\}$.

(6)                $y_{2n} = Sx_{2n} = A_{2n} x_{2n-1}$  for every  $n \in N$.

Let  $d_n = d(y_n, y_{n+1})$  for every $n \in N$. Consequently the following two lemmas also hold.

**Lemma 1.2** :  The  sequence $\{d_n\}_{n \in N}$ converges  to  zero.

**Proof :**  Using (4), we have for $n \in N_o$.

$ad(A_{2n+1} x_{2n}, A_{2n+2} x_{2n+1}) + bd(Tx_{2n+1}, A_{2n+2} x_{2n+1}) - \min \{d(Tx_{2n+1}, A_{2n+1} x_{2n}), d(Sx_{2n}, A_{2n+1} x_{2n})\}$

$$\leq f( \max\{d(Sx_{2n},Tx_{2n+1}), d(Sx_{2n}, A_{2n+1}x_{2n}), d(Tx_{2n+1}, A_{2n+2}x_{2n+1}), d(Tx_{2n+1}, A_{2n+1}x_{2n}),$$
$$[d(Sx_{2n}, A_{2n+2}x_{2n+1})+ d(Tx_{2n+1}, A_{2n+1}x_{2n})/2\})$$

i.e.,   $ad_{2n+1} + bd_{2n+1} - \min\{0,d_{2n}\} \leq f(\max\{d_{2n}, d_{2n}, d_{2n+1}, 0, (d_{2n}+ d_{2n+1})/2\})$

i.e.,                 $(a+b) d_{2n+1} \leq f(\max\{d_{2n}, d_{2n+1}, (d_{2n}+d_{2n+1})/2\})$

We claim that $d_{2n+1}< d_{2n}$. If $d_{2n+1}> d_{2n}$ for some $n \in N_o$, then $(a+b) d_{2n+1} \leq f(d_{2n+1})$

i.e.,           $d_{2n+1} \leq f(d_{2n+1})/(a+b) < f(d_{2n+1})$, a contradiction to the property of f. Thus we get

(7)    $d_{2n+1} \leq d_{2n}$ and $(a+b) d_{2n+1} \leq f(d_{2n})$ i.e., $d_{2n+1} \leq f(d_{2n})/(a+b) \leq f(d_{2n})$     for every $n \in N_o$.

Similarly, we can show that

(8)                 $d_{2n+2} \leq d_{2n+1}$ and $d_{2n+2} \leq f(d_{2n+1})$     for every $n \in N_o$.

From (7) and (8) we find that $\{d_n\}_{n \in N}$ is a decreasing sequence such that $d_{n+1} \leq f(d_n)$ for every $n \in N_o$. Then by Lemma 1.1, the sequence $\{d_n\}_{n \in N}$ converges to 0.


**Lamma 1.3** The sequences $\{y_{2n}\}_{n \in N}$ and $\{y_{2n+1}\}_{n \in No}$ are Cauchy sequences.

**Proof :** Suppose it is no so, then there exists an $\varepsilon > 0$ such that for any integer 2k there exists two sequences $\{2m(k)\}_{k \in N}$ and $\{2n(k)\}_{k \in N}$ with $2m(k) > 2n(k) > k$ for which

(9)                                   $d(y_{2n(k)},y_{2m(k)}) > \varepsilon.$

If $2m(k)$ denotes the smallest integer exceeding $2n(k)$ satisfying (9), we have, $d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon.$

Then,                 $\varepsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}$

For any even integer 2k this implies that

(10)                                   $\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon.$

Using (4) and the non decreasing property of f, we get

$$d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)})$$
$$\leq d_{2n(k)} - b\, d_{2m(k)-1} + \min\{d(y_{2m(k)-1}, y_{2n(k)+1}), d_{2n(k)}\}$$
$$+ f(\max\{d(y_{2n(k)}, y_{2m(k)-1}), d_{2n(k)}, d_{2m(k)-1}, d(y_{2m(k)-1}, y_{2n(k)+1}),$$
$$[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)-1}, y_{2n(k)+1})]/2\})$$
$$\leq d_{2n(k)} - b\, d_{2m(k)-1} + \min\{d(y_{2m(k)-1}, y_{2n(k)+1}), d_{2n(k)}\}$$
$$+ f(\max\{d(y_{2n(k)}, y_{2m(k)}) + d_{2m(k)-1}, d_{2n(k)}, d_{2m(k)-1}, d_{2m(k)-2}+ \varepsilon + d_{2n(k)},$$
$$[d(y_{2n(k)}, y_{2m(k)}) + d_{2m(k)-2} + \varepsilon + d_{2n(k)}/2\})$$

Using Lemma 1.2, (10) and the upper semi continuity of f, we get as $k \to \infty$,

$$\varepsilon \leq f(\max\{\varepsilon, 0, 0, \varepsilon, [\varepsilon + \varepsilon]/2\}) = f(\varepsilon)$$

which contradicts the fact that $f(\varepsilon) < \varepsilon$ as $\varepsilon > 0$. Thus $\{y_{2n}\}_{n \in N}$ is a Cauchy sequence. Similarly we can show that $\{y_{2n+1}\}_{n \in N}$ is also a Cauchy sequence. Hence the lemma.

## 2.   Main Results

Taking a clue from the result of Sessa et. al. [10], we have the following theorem for a family of mappings.

**Theorem 2.1** Let $S, T : X \to X$ and either S or T be continuous. Then S and T have a common fixed point w if and only if there exists a family $\{A_i\}_{i \in N}$ of self mappings of X and $f \in F$ satisfying the condition

(vii)      $A_i(X) \subset S(X) \cap T(X)$, for each $i \in N$

(viii)      $A_i$ weakly commutes with S and T, for each $i \in N$

(ix)      $ad(A_ix, A_jy) + bd(Ty, A_jy) - \min\{d(Ty,A_ix), d(Sx, A_ix)\}$

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$$\leq f\,(\max\,\{d(Sx,Ty),\,d(Sx,A_ix),\,d(Ty,A_jy),\,d(Ty,A_ix),\,[d(Sx,A_jy)+d(Ty,A_ix)]/2\})$$

for all x, y $\in$ X, i, j $\in$ N,  i $\neq$ j, a, b $\geq$ 0 with a $\geq$ 2. Then w is the unique common fixed point of $\{A_i\}_{i\in N}$, S and T.

**Proof :**  By  Lemma 1.3, the sequence $\{y_{2n+1}\}_{n\in N_o}$ , as defined  in (5)  converges to a point  w, for X is complete.  By Lemma 1.2 ,  we have

$$0 = \lim_{n\to\infty} d_{2n} = \lim_{n\to\infty} d(y_{2n},\,y_{2n+1}) = \lim_{n\to\infty} d(y_{2n},w).$$

Thus the sequence $\{y_{2n}\}$, defined in (6), also converges to w. Now we suppose the continuity of the mapping S. Then

$$Sw = S(\lim_n y_{2n+1}) = S(\lim_{n\to\infty} A_{2n+1}\,x_{2n}) = \lim_{n\to\infty}SA_{2n+1}\,x_{2n}$$

Since  S weakly commutes with  $A_i$  for  any i $\in$ N,  we have

$$d(A_{2n+1}\,Sx_{2n},\,Sw) \leq d\,(A_{2n+1}\,Sx_{2n},\,SA_{2n+1}\,x_{2n}) + d(SA_{2n+1}x_{2n},\,Sw)$$

$$\leq d(A_{2n+1}\,x_{2n},\,Sx_{2n}) + d(SAx_{2n+1},\,Sw)$$

which implies, as  n$\to \infty$, that the sequence $\{A_{2n+1}\,Sx_{2n}\}_{n\in N}$   also converge to  Sw.

using (vii), we deduce

$$ad(A_{2n+1}\,Sx_{2n},\,A_{2n}x_{2n-1}) + bd(Tx_{2n-1},A_{2n}x_{2n-1}) - \min\,\{d(Tx_{2n-1},\,A_{2n+1}Sx_{2n}),\,d(S^2x_{2n},\,A_{2n+1}Sx_{2n})\}$$

$$\leq f\,(\max\,\{d(S^2x_{2n},Tx_{2n-1}),\;d(S^2x_{2n},\,A_{2n+1}\,Sx_{2n}),\,d(Tx_{2n-1},\,A_{2n}x_{2n-1}),$$

$$d(Tx_{2n-1},A_{2n+1}\,Sx_{2n}),\,[d(S^2x_{2n},\,A_{2n}x_{2n-1})+d(Tx_{2n-1},A_{2n+1}\,Sx_{2n})]/2\})$$

which  implies, as limit n $\to \infty$ , that

$$ad(Sw,w) \leq f\,(\max\,\{d(Sw,w),\,d(Sw,Sw),\,d(w,w),\,d(w,Sw),\,[d(Sw,w) + d(w,Sw)/2\})$$

$$= f(d(Sw,w)) < d(Sw,w)\ .$$

Therefore,   Sw = w  .

Using  (vii) again, we have  for any odd  integer  i $\in$ N.

$$ad(A_i\,w,A_{2n}x_{2n-1}) + bd(Tx_{2n-1},\,A_{2n}x_{2n-1}) - \min\,\{\,d(Tx_{2n-1},\,A_iw),d(Sw,A_i\,w)\}$$

$$\leq f\,(\max\{d(Sw,Tx_{2n-1}),\,d(Sw,\,A_iw),\,d(Tx_{2n-1},\,A_{2n}\,x_{2n-1}),\,d(Tx_{2n-1},\,A_iw),$$

$$[d(\,Sw,\,A_{2n}x_{2n-1})+d(Tx_{2n-1},\,A_iw)]/2\})$$

Taking limit as  n $\to \infty$, we  have

$$ad(\,A_i\,w,\,w) + bd(w,w) - \min\{d(w,\,A_iw)\,,d(Sw,\,A_i\,w)\}$$

$$\leq f\,(\max\{d(Sw,w),\,d(Sw,A_iw),\,d(w,w),\,d(w,A_iw),\,[d(Sw,w) + d(w,A_iw)/2\})$$

or,        $ad(A_iw,\,w) - d(A_iw,\,w) \leq f(\max\{0,\,d(w,\,A_iw),\,0,\,d(w,\,A_iw),\,d(w,\,A_iw)/2\})$

or,            $(a\text{-}1)\,d(A_i\,w,\,w)\ \leq f(d(A_i\,w,w)) < d(A_i\,w,w)$

which gives $w = A_i w$, for any odd integer $i \in N$ as $a \geq 2$.

Now by (vii) we clearly have $w \in \cap_{n \in N} A_{2n+1}(X) \subseteq T(X)$ and therefore there exists a point $w' \in X$ such that $A_i w = Tw' = w$ for any odd integer $i \in N$.

Then using (vii) we deduce for any even integer $j \in N$ and for any odd integer $i \in N$.

$$ad(A_i w, A_j w') + bd(Tw', A_j w') - \min \{d(Tw', A_i w), d(Sw, A_i w)\}$$

$$\leq f \max\{d(Sw, Tw'), d(Sw, A_i w), d(Tw', A_j w'), d(Tw', A_i w), [d(Sw, A_j w') + d(Tw', A_i w)]/2\}$$

or,     $ad(w, A_j w') + bd(w, A_j w') - \min \{d(w,w), d(w,w)\}$

$$\leq f \max \{d(w,w), d(w,w), d(w, A_j w'), d(w,w), [d(w, A_j w') + d(w,w)]/2\}$$

Hence        $(a+ b) d(w, A_j w') \leq f(d(w, A_j w')) < d(w, A_j w')$,

which implies that $A_j w' = w$, for any even integer $j \in N$. $A_k$ being weakly commuting with T for any $k \in N$, we have

$$d( TA_j w', A_j Tw') \leq d (Tw', A_j w') = d(w,w) = 0$$

and hence

$$Tw = TA_j w' = A_j Tw' = A_j w.$$

for any even integer $j \in N$. Moreover (viii) implies for any odd integer $i \in N$ and for any even integer $j \in N$.

  $ad (A_i w, A_j w) + bd(Tw, A_j w) - \min \{ d(Tw, A_i w), d(Sw, A_i w)\}$

   $\leq f(\max \{d(Sw,Tw), d(Sw, A_i w), d(Tw, A_j w), d(Tw, A_i w), [d(Sw, A_j w) + d(Tw, A_i w)]/2\})$

or,    $ad(w, Tw) + bd(Tw, Tw) - \min \{d(Tw, w), d(w,w)\}$

$$\leq f(\max\{d(Sw, Tw), d(Sw, A_i w), d(Tw, A_j w), d(Tw, A_i w), [d(Sw, A_j w) + d(Tw, A_i w)]/2\})$$

Thus,                $ad (w, Tw) \leq f (\max\{d(w, Tw), 0, 0, d(Tw, w), d(w, Tw)\})$

 i.e.                    $a\, d(w, Tw) \leq f(d(w, Tw)) < d(w, Tw)$.

Thus $w = Tw$ and so $A_j w = w$ for any even integer $j \in N$. Therefore $w$ is a common fixed point of $\{A_i\}_{i \in N}$, $S$ and $T$.

Similar proof can be exhibited if one supposes the continuity of T instead of S. Further from (vii) it is easily seen that $w$ is unique common fixed point of $\{A_i\}_{i \in N}$, S and T. This concludes the sufficient part of the proof.

To show the necessary of the condition, let $w$ be a common fixed point of S and T. Define

$A_i (x) = w$ for any $i \in N$ and for any $x \in X$. Now since $d(A_i x, A_j y) = d(w,w) = 0$ for any

$i,j \in N$, $i \neq j$ and for any $x,y \in X$. It is trivial that (x) holds for any function $f \in F$ and further (xiv) is verified being

$$\{w\} = A_i(x) \subseteq S(X) \cap T(X)$$

for any $i \in N$. Clearly w is the unique common fixed point of $\{A_i\}_{i \in N}$, S and T. This completes the proof.

□

Using weak commutativity of mappings we have an extension of Theorem 1.2 Chang [2] as follows.

**Theorem 2.2** Let S, T : $X \to X$ and either S or T be continuous. Then S and T have a common fixed point w if and only if there exists two self mappings A,B of X and a function $f \in F$ such that

(x)  $A(X) \cup B(X) \subseteq S(X) \cap T(X)$.

(xi)  both A and B weakly commute with S and T .

(xii)  $d(Ax,By) \leq f(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), [d(Sx,By)+ d(Ty,Ax)]/2\})$

Further w is the unique common fixed point of A,B, S and T.

As the proof goes in a similar manner as of Theorem 2.1, so we omit it. However we give an example in order to show that the Theorem 2.2 is stronger result than Theorems 1.2.

**Example 2.1**    Let X = [0,1] with Euclidean metric.

Define    $Ax = \frac{x}{8} - \frac{x^2}{64}$,  $Bx = \frac{x}{4} - \frac{x^2}{16}$, $Sx = \frac{x}{2}$, $Tx = x$ for any $x \in X$.

we have    $ASx = A(\frac{x}{2}) = \frac{x}{16} - \frac{x^2}{256}$ and $SAx = S(\frac{x}{8} - \frac{x^2}{64}) = \frac{x}{16} - \frac{x^2}{128}$

$$d(ASx, SAx) = (\frac{x}{16} - \frac{x^2}{256} - \frac{x}{16} + \frac{x^2}{128})$$

$$= \frac{x^2}{256} \leq \frac{3x}{8} + \frac{x^2}{64} = \frac{x}{2} - (\frac{x}{8} - \frac{x^2}{64}) = d(Sx, Ax) \quad \text{for any } x \in X.$$

Thus S weakly commute with A, but S and A do not commute being

$$ASx = \frac{x}{16} - \frac{x^2}{256} \neq \frac{x}{16} - \frac{x^2}{128} = SAx \text{ for any } x \in (0, 1].$$

Moreover B commutes with T. Let F be the set of all real function f: $[0, \infty) \to [0, \infty)$ such that for every $f \in F$ is non decreasing, upper semi continuous and $f(t) < t$ for any $t > 0$ and let

$$f(t) = \frac{1}{4}t - \frac{1}{16}t^2 \quad \text{if } 0 \leq t \leq 1$$

$$= \frac{3}{8}t \qquad \text{if } t > 1.$$

Then we have

$$d(Ax, By) = |\frac{x}{8} - \frac{x^2}{64} - \frac{y}{4} + \frac{y^2}{16}|$$

$$= \left| \left( \tfrac{x}{6} - \tfrac{y}{4} \right) - \tfrac{1}{16} \left( \tfrac{1}{4}x^2 - y^2 \right) \right|$$

$$= \tfrac{1}{4} \left| \tfrac{1}{2} x - y \right| \left[ 1 - \tfrac{1}{4} \left( \tfrac{1}{2}x + y \right) \right]$$

$$\le \tfrac{1}{4} \left| \tfrac{1}{2} x - y \right| \left[ 1 - \tfrac{1}{4} \left| \tfrac{1}{2} x - y \right| \right]$$

$$\le \tfrac{1}{4} \left| \tfrac{1}{2} x - y \right| - \tfrac{1}{16} \left| \tfrac{1}{2} x - y \right|^2$$

$$= f(d(Sx, Ty))$$

$$\le f(\max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{1}{2}[d(Sx,By) + d(Ty,Ax)]\})$$

for any  x,y $\in$ X . Further,

$$A(X) \cup B(X) = [0, \tfrac{7}{64}] \cup [0, \tfrac{3}{16}]$$

$$= [0, \tfrac{3}{16}] \subset [0, \tfrac{1}{2}] = [0, \tfrac{1}{2}] \cap [0,1]$$

$$= S(X) \cap T(X) .$$

Thus we see that all the assumptions of the Theorem 2.2 are verified and 0 is the unique common fixed point, whereas Theorems 1.1 and 1.2 are not applicable because the mappings S and A do not commute.

**Theorem  2.3**  Let $T_1$ and $T_2$ be two continuous mappings of a complete metric space (X, d)  into itself such that

$$d( T_1 x, T_2 y) < \max \{d(x,y), d(x,T_1 x)\}$$

and there exists a subset A $\subset$ X and a point $x_0 \in$ A satisfying  the following conditions :

(xiii)     $\qquad\qquad\qquad d(x_0, T_i x) - d(T_1 x_0, T_1 T_2\, x) \ge 2 d(x_0, T_1 x_0),$

for x, y in X − A, i = 1,2 and $T_1 T_2 = T_2 T_1.$

(xiv)    $\qquad\qquad\qquad d(T_1 x, T_2 y) \le \alpha\, d(x, y)\{d(x, T_1 x)\, d(y, T_2 y)\}^{1/2}$

for x, y in A, where $\alpha$ is a monotonically decreasing function from $[0, \infty)$ into $[0, 1)$. Then there exists a unique common fixed point of $T_1$ and $T_2$.

**Proof** : Suppose that  $x_0 \ne T_1 x_0$  and define a sequence $\{x_n\}$ of elements $x_n \in$ X,

$$T_1 x_0 = x_1,\ T_2 x_1 = x_2, \ldots,\ T_1\, x_{2n} = x_{2n+1},\ T_2\, x_{2n+1} = x_{2n+2}.$$

we have    $\qquad\qquad d(T_1\, x, T_2\, y) < \max \{d(x, y), d(x, T_1 x)\}$

Hence,    $\qquad\qquad d(x_0, x_{2n+1}) < \max \{d(x_0, x_1), d(x_0, x_0)\} = d(x_0, x_1).$

From the triangle inequality and  $T_1 T_2 = T_2 T_1$  we get,

$$d(x_0, x_{2n+1}) \le d(x_0, x_1) + d(x_1, x_{2n+2}) + d(x_{2n+2},\, x_{2n+1})$$

$$= d(x_0, x_1) + d(T_1 x_0, T_2 T_1 x_{2n}) + d(x_{2n+2},\, x_{2n+1})$$

$$< 2d(x_0, x_1) + d(T_1 x_0, T_2T_1 x_{2n})$$

Thus,

$$d(x_0, T_1x_{2n}) - d(T_1x_0, T_2T_1x_{2n}) < 2d(x_0, T_1x_0).$$

Hence from the condition (xx), it follows that $x_{2n+1} \in A$ for every n.

Similarly $x_{2n+2} \in A$ for all n. Therefore $x_n \in A$ for each n.

Next we show that the sequence $\{x_n\}$ is bounded. For this consider,

$$d(x_0, x_{2(n+1)}) \le d(x_0, T_1x_0) + d(T_1x_0, T_2x_{2n-1}) + d(T_1x_{2n}, T_2x_{2n-1}) + d(x_{2n+1}, x_{2n+2})$$

$$\le 3\, d(x_0, T_1x_0) + \alpha\, (d(x_0, x_{2n-1}))\{\, d(x_0, x_1)\, d(x_{2n-1}, x_{2n})\}^{1/2}$$

$$\le \{3 + \alpha\, d(x_0, x_{2n-1})\}\, d(x_0, x_1)$$

Hence for a given $d_0 > 0$ with $d(x_0, x_{2n-1}) \ge d_0$, we get

$$d(x_0, x_{2n+2}) \le \{\, 3 + \alpha\, d_0\}\, d(x_0, x_1).$$

Similarly we can show that

$$d(x_0, x_{2n+1}) \le \{\, 2 + \alpha\, d_0^*\}\, d(x_0, x_1),\ \text{for given } d_0^* > 0,\ d(x_0, x_{2n+1}) \ge d_0^*$$

and hence $\{x_n\}$ is bounded. By routine calculation, it follows that for each n,

$$d(x_n, x_{n+1}) \le \{\beta(d(x_{n-1}, x_n)), \beta(d(x_{n-2}, x_{n-1})),\dots, \beta(d(x_0, x_1))\}\, d(x_0, x_1),\ \text{where } \beta = \alpha^2.$$

Let $\varepsilon > 0$ .  If $d(x_i, x_{i+1}) \ge \varepsilon$ for $i = 0,1,2,..$, then

$$\beta\, (d(x_i, x_{i+1})) \le \beta\, (\varepsilon)$$

for $i = 0,1,2,\dots$ and also $0 \le \beta(t) < 1$. Therefore

$$d(x_n, x_{n+1}) \le (\beta(\varepsilon))^n\, d(x_0, x_1).$$

This proves that $\{x_n\}$ is a Cauchy sequence.  As X is complete, $\lim_{n\to\infty} x_n = \xi \in X$. Using continuity of $T_1$ and $T_2$, we find that $\xi$ is a common fixed point of $T_1$ and $T_2$. Uniqueness of $\xi$ is obvious.    □

Putting $A = X$ in Theorem 2.3, we get the following corollary.

**Corollary 2.1**   If X is a complete metric space and $T_1, T_2$ satisfy

$$d(T_1x, T_2y) \le \alpha\, d(x, y)\{d(x, T_1x)\, d(y, T_2y)\}^{1/2}$$

for all x, y in X . Then $T_1$ and $T_2$ have a unique common fixed point.

**Remark 2.1**  The inequality condition (4) is more general than the inequality condition of Chang [2] and Sessa et. al. [10] and only one of S and T is continuous than that in Chang [2], where both the mappings were continuous. So our result generalizes the result of Chang in two ways.

**References :**

[1]  Shih-sen Chang, Math. Japonica, 26, p121 **(1981).**

[2]  _____, Math. Japonica, 29, p527 **(1984).**

[3]  B. Fisher, Bull. Inst. Math. Acad. Sinica, 9, p399 **(1981).**

[4]  O. Hadzic, Math. Japonica, 29, p287 **(1984).**

[5]  S. A. Hussain and V. M. Sehgal, Math. Japonica, 26, p287 **(1981).**

[6]  K. Iseki, Bull. Austral. Math. Soc., 10, p75 **(1974).**

[7]  M. S. Khan and M. Imdad, Glasnik Math.,18 (38), p321 **(1983).**

[8]  _____,  Comm. Fac. Sci. Univ. Ankara, 32, p105 **(1983).**

[9]  B. A. Meade, B.A and Singh, S.P., Bull.Aus.Math.Soc., 16, p49 **(1977).**

[10] S. Sessa R.N. Mukherjee and T. Som, Math. Japonica, 31(2), p235 **(1986).**

[11] S. L. Singh, Math. Sem. Notes Kobe Univ., 5, p131 **(1977).**

[12] S. L. Singh and B. M. L. Tiwari, Proc. Nat. Acad. Sci. Indian Sec. A 51(I), p41**(1981).**