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Fixed Point and Generalized Hyers-Ulam-Rassias Stability of a Quadratic Functional Equation

Ehsan Movahednia

Behbahan Khatam Al-Anbia University of Technology, Behbahan- Iran

E-mail: ***ehsar.m@gmail.com***

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Abstract

In this paper, using the fixed point alternative approach, we investigate the Hyers Ulam-Rassias stability of the following functional equation

$$f(2x-yi)+f(x-2yi)=4f(x-yi)+f(x)-f(y)$$

in Banach spaces.

Keywords: Fixed point theory ,Hyers-Ulam-Rassias stability.

1. Introduction

The stability of functional equations appeared at first by Ulam in 1940 [1], where in 1941, Hyers studied a version of this problem in [2].In 1978, Th.M.Rassias [3] extended the result of Hyers as follows:

Suppose that E_1, E_2 *are Banach spaces and* $f: E_1 \to E_2$ *is a mapping for which there exist* $\varepsilon > 0$ *and* $0 \le p < 1$ *such that* $\| f(x+y)-f(x)-f(y)\| < \epsilon\left(\| x\|^p + \| y\|^p\right)$ *for all* $x, y \in E_1$*, then there exists a unique additive mapping* $T: E_1 \to E_2$ *such that* $\| f(x)-T(x)\| \le \frac{2\epsilon}{2-2p}\| x\|^p$ *for all* $x \in E_1$.

Subsequently the result of Hyers was generalized by Aoki [4] for additive mapping and by Th. M. Rassias [5] for linear mapping by considering an unbounded cauchy difference. The paper of Th. M. Rassias [5] has provided a lot of influence in the development of what we call the Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

a dellac si*quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f:X\to Y$,where X is a normed space and Y is a Banach space.

G. Isac and Th. M. Rassias [7] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors(see [8],[9],[10]).

hT. M. Rassias and Choonkil Park in [11] proved the Hyers-Ulam stability of the functional equation in Banach Spaces.

2. Preliminaries

Theorem 2.1. [12] A function $f:X\to Y$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.1}$$

if and only if $f:X\to Y$ satisfies the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y) \tag{2.2}$$

Therefore, every solution of functional equation (2.2) is quadratic function.

Theorem 2.2. [13] Let X,Y be vector spaces. If $f:X\to Y$ satisfies

$$f(x+iy)+f(x-iy)=2f(x)-2f(y) \tag{2.3}$$

for all $x,y\in X$, then the mapping $f:X\to Y$ satisfies (2.1). If $f:X\to Y$ satisfies (2.1) such that $f(iy)=-f(y)$ then f satisfies (2.3).

Corollary 2.1. teL X,Y be vector spaces. If $f:X\to Y$ satisfies

$$f(2x-yi)+f(x-2yi)=4f(x-yi)+f(x)-f(y) \tag{2.4}$$

and $f(iy)=-f(y)$ for all $x,y\in X$ then the mapping $f:X\to Y$ is quadratic.

Definition 2.1. Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following conditions

(a) $d(x,y)=0$ if and only if $x=y$ for all $x,y\in X$;

(b) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(c) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 2.3. Let (X , d) be a complete generalized metric space and $J:X\to X$. be a strictly contractive mapping with Lipschitz constant $L<1$. Then, for all $x\in X$, either

$$d(J^n x, J^{n+1}x)=\infty \tag{2.5}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;

(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(c) y^* is the unique fixed point of J in the set $\mathrm{Y} = \{y \in \mathrm{X} : d(J^{n_0}x, y) < \infty\}$;

(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \mathrm{Y}$.}

3. Main Results

Throughout this section, using fixed point method, we prove the Hyers-Ulam-Rassias stability of functional equation.

In the rest of the paper, assume that X is a normed vector space and Y is a Banach space.

Theorem 3.1. Let $f : \mathrm{X} \to \mathrm{Y}$ be a mapping with $f(ix) = -f(x)$ and $f(0) = 0$ for all $x \in \mathrm{X}$ for which there exists a function $\varphi : \mathrm{X} \times \mathrm{X} \to [0, \infty)$ and $0 < L < 1$ such that

$$\varphi(x, ix) \leq 3^2 L\varphi\left(\frac{x}{3}, \frac{ix}{3}\right) \text{(3.6)}$$

and

$$\| f(2x - yi) + f(x - 2yi) - 4f(x - yi) - f(x) + f(y) \| \leq \varphi(x, y) \tag{3.8}$$

for all $x, y \in \mathrm{X}$. Then there exists a unique quadratic mapping $Q : \mathrm{X} \to \mathrm{Y}$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{9 - 9L}\left[\frac{1}{2}\varphi(x, ix) + 2\varphi(x, 0)\right] \tag{3.8}$$

for all $x, y \in \mathrm{X}$.

Proof: Let consider the set $\Delta := \{h : \mathrm{X} \to \mathrm{Y} \mid h(0) = 0\}$ and the mapping d define on $\Delta \times \Delta$ by

$$d(g, h) := \inf\left\{\varepsilon \in (0, \infty) : \| g(x) - h(x) \| \leq \varepsilon\left(\frac{\varphi(x, ix)}{2} + 2\varphi(x, 0)\right), \text{for all }, x \in \mathrm{X}\right\}$$

where $inf \varnothing = +\infty$. It is easy to show that (Δ, d) is a complete metric space (13). Let consider the mapping

$$J : \Delta \to \Delta \quad Jg(x) = 3^{-2} g(3x) \text{ for all } x \in \mathrm{X}.$$

Fix a $\varepsilon \in (0, \infty)$ and take $g, h \in \Delta$ such that $d(g, h) < \varepsilon$. By the definitions of d and J, we have

$$\left\|3^{-2} g(3x) - 3^{-2} h(3x)\right\| \leq \frac{\varepsilon}{3^2}\left(\frac{\varphi(3x, i3x)}{2} + 2\varphi(3x, 0)\right) \quad \text{for all } x \in \mathrm{X}$$

so by (3.6), we have

$$\left\|3^{-2} g(3x) - 3^{-2} h(3x)\right\| \leq \varepsilon L\left(\frac{1}{2}\varphi(x, ix) + 2\varphi(x, 0)\right) \text{ for all } x \in \mathrm{X}.$$

This implies that

$$d(g, h) < \varepsilon \to d(Jg, Jh) \leq L\varepsilon\, d(g, h)$$

for all $g, h \in \Delta$. On the other hand, replacing y by ix in (3.7), we obtain

$$\| f(3x)-2f(2x)-f(x)\| \le \frac{1}{2}\varphi(x,ix). \qquad (3.9)$$

for all $x \in X$. Also, replacing y by 0 in (3.7), we have

$$\| f(2x)-4f(x)\| \le \varphi(x,0). \qquad (3.10)$$

for all $x \in X$. Combining (3.9), (3.10) and using triangular inequality, we get

$$\begin{aligned} \| f(3x)-3^2 f(x)\| &= \| f(3x)-2f(2x)-f(x)+2(f(2x)-4f(x))\| \\ &\le \| f(3x)-2f(2x)-f(x)\| + 2\| f(2x)-4f(x)\| \\ &\le \frac{\varphi(x,ix)}{2}+2\varphi(x,0) \qquad (3.11) \end{aligned}$$

lla rof $x \in X$ .Therefore

$$\left\| 3^{-2} f(3x)-f(x)\right\| \le \frac{1}{9}\left(\frac{1}{2}\varphi(x,ix)+2\varphi(x,0)\right) \qquad (3.12)$$

for all $x \in X$. This means that

$$d(Jf, f) \le \frac{1}{9} < \infty.$$

By Theorem 2.3, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J , i.e.,

$$Q(3x) = 3^2 Q(x) \qquad (3.13)$$

for all $x \in X$.

The mapping Q is a unique fixed point of J in the set $M = \{g \in S : d(h,g) < \infty\}$. This implies that Q is a unique mapping satisfying (3.13) such that there exists a $\mu \in (0,\infty)$

satisfying

$$\| f(x)-Q(x)\|_Y \le \mu\left(\frac{1}{2}\varphi(x,ix)+2\varphi(x,0)\right)$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{3^{2n}} f\left(3^n x\right) = Q(x) \qquad (3.14)$$

for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f,Q) \le \frac{d(f,Jf)}{1-L} \le \frac{1}{9-9L}.$$

This implies that the inequalities (3.8) holds. Replacing x, y by $3^n x, 3^n y$ in (3.7) ,we have

$$\| Q(2x-yi)+Q(x-2yi)-4Q(x-yi)-Q(x)+Q(y)\|$$

$$\leq \lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y)}{3^{2n}} = 0 \tag{3.15}$$

So

$$Q(2x - yi) + Q(x - 2yi) = 4Q(x - yi) + Q(x) - Q(y).$$

Therefore Q is a quadratic mapping, this completes the proof. □

Theorem 3.2. Assume that $\varphi : X \times X \to [0, \infty)$ be a function such that, for which $0 < L < 1$

$$\varphi\left(\frac{x}{3}, \frac{ix}{3}\right) \leq \frac{L\varphi(x, ix)}{3^2} \tag{3.16}$$

and $f : X \to Y$ be a mapping with $f(0) = 0$ and $f(iy) = -f(y)$ satisfying (3.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \leq \frac{L}{9 - 9L}\left[\frac{\varphi(x, ix)}{2} + 2\varphi(x, 0)\right] \tag{3.17}$$

for all $x \in X$.

Proof: Replacing x by $\frac{x}{3}$ in (3.11), we obtain

$$\left\| f(x) - 3^2 f\left(\frac{x}{3}\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{3}, \frac{ix}{3}\right) + 2\varphi\left(\frac{x}{3}, 0\right)$$

$$\leq \frac{L}{3^2}\left[\frac{\varphi(x, ix)}{2} + 2\varphi(x, 0)\right] \tag{3.18}$$

lla rof $x \in X$ teL . (Δ, d) be the generalized metric space defined in the proof of Theorem 2.3 gnippam raenil a redisnoC $J : \Delta \to \Delta$ such that

$$Jh(x) := 3^2 h\left(\frac{x}{3}\right) \tag{3.19}$$

for all $x \in X$ .Let $g, h \in \Delta$ be such that $d(g, h) =$ ònehT .

$$\| g(x) - h(x) \| \leq ò\left[\frac{\varphi(x, ix)}{2} + 2\varphi(x, 0)\right]$$

lla rof $x \in X$ and so

$$\| Jg(x) - Jh(x) \| = \left\| 3^2 g\left(\frac{x}{3}\right) - 3^2 h\left(\frac{x}{3}\right)\right\| \leq 3^2 ò\left[\frac{1}{2}\varphi\left(\frac{x}{3}, \frac{ix}{3}\right) + 2\varphi\left(\frac{x}{3}, 0\right)\right]$$

$$\leq 3^2 ò\frac{L}{3^2}\left[\frac{\varphi(x, ix)}{2} + 2\varphi(x, 0)\right]$$

lla rof $x \in X$ .Thus $d(g, h) =$ ò implies that $d(Jg, Jh) \leq L$ò .This means that

$$d(Jg, Jh) \leq L d(g, h)$$

lla rof $g, h \in \Delta$ .It follows from (3.18) that

$$d(f, Jf) \leq \frac{L}{9} < +\infty. \tag{3.20}$$

There exists a mapping $Q : X \to Y$ satisfying the following:

-1 Q is a fixed point of J ,that is,

$$Q\left(\frac{x}{3}\right)=\frac{1}{9}Q(x) \tag{3.21}$$

for all $x \in X$ .The mapping Q is a unique fixed point of J in the set

$$\Omega=\{h \in S : d(g,h)<\infty\}.$$

taht seilpmi sihT Q is a unique mapping satisfying (3.21) such that there exists $\mu \in (0,\infty)$ satisfying

$$\| f(x)-Q(x) \| \leq \mu\left[\frac{\varphi(x,ix)}{2}+2\varphi(x,0)\right]$$

lla rof $x \in X$.

-2 $d(J^n f,Q) \to 0$ as $n \to \infty$ .This implies the equality

$$\lim_{n\to\infty} 3^{2n} f\left(\frac{x}{3^n}\right)=Q(x) \tag{3.22}$$

lla rof $x \in X$.

-3 $d(f,Q) \leq \dfrac{d(f,Jf)}{1-L}$ with $f \in \Omega$,which implies the inequality

$$d(f,Q) \leq \frac{L}{9-9L}. \tag{3.23}$$

The rest of the proof is similar to the proof of Theorem 3.1. This completes the proof. □

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