Solving Fractional Partial Differential Equation by Using Wavelet Operational Method

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Abstract

In this paper, we use a method based on the operational matrices to the solution of the fractional partial differential equations. The main approach is based on the operational matrices of the Haar wavelets to obtain the algebraic equations. The fractional derivatives are described in Caputo sense. Some examples are included to demonstrate the validity and applicability of the techniques.

1. Introduction

In recent years, fractional calculus is one of the interest issues that attract many scientists, specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena by boundary value problems of fractional deferential equations [1-4]. For achieve extra information in fractional calculus, reader can refer to more valuable books that are written by authors [5-9]. Many physical phenomena in areas such as damping laws, diffusion processes, etc, can be modeled with partial differential equations of fractional order. Many authors developed wavelet-based numerical solutions of partial differential equations [10-16].

Useful applications of wavelet operational matrix for numerical solutions of differential equations can be found [17-23]. This issue with characteristic of wavelet functions is motivations for using them to find the solutions of differential equations of fractional order and mostly fractional partial differential equations.

In this paper, we use a method based on the operational matrices to the solution of the fractional partial differential equations. The main approach is based on the operational matrices of the Haar wavelets to obtain the algebraic equations.

2. Haar wavelets operational matrixes

The operational matrix of an orthogonal matrix $\Phi(t)$, F_Φ, can be expressed by

$$F_\Phi = \Phi \cdot F_B \cdot \Phi^{-1} \tag{2.1}$$

where F_B is the operational matrix of the block pulse function

$$
F_{Bm} = \frac{1}{m}
\begin{bmatrix}
\frac{1}{2} & 1 & . & . & . & 1 \\
0 & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & 1 & . \\
. & . & . & 0 & \frac{1}{2} & 1 \\
0 & . & . & . & 0 & \frac{1}{2}
\end{bmatrix} . \tag{2.2}
$$

If the transform matrix Φ is unitary, that is $\Phi^{-1} = \Phi^T$ then (2.1) can be rewritten as:

$$F_\Phi = \Phi \cdot F_B \cdot \Phi^T \tag{2.3}$$

The approach is simple and computer oriented, therefore very useful in practice. Wavelets have become an increasingly popular tool in the computational sciences. They have numerous applications in a wide range of areas such as signal analysis, data compression and many others. The Haar wavelets have the following features:(1) highly energy packing; (2) the base functions are consisted of three simple integers, 0 and -1, and 1 only. The properties are useful in speeding up the computation. So we will use the Haar wavelets and its operational matrix for demonstration throughout this paper. Let us begin by briey reviewing the Haar functions [24,25]. The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1) by

$$h_0(t) = \frac{1}{\sqrt{m}}$$

$$
h_i(t) = \frac{1}{\sqrt{m}}
\begin{cases}
2^{\frac{j}{2}} & \frac{k-1}{2^j} \leq t \leq \frac{k-\frac{1}{2}}{2^j} \\
-2^{\frac{j}{2}} & \frac{k-\frac{1}{2}}{2^j} \leq t \leq \frac{k}{2^j} \\
0 & otherwise
\end{cases} \tag{2.4}
$$

where $i = 1, 2, \ldots m-1$, $m = 2^r$ and r is a positive integer. j **and** k represent the integer decomposition of the index i, i.e. $i = 2j + k - 1$. Theoretically, this set of functions is complete [24]. In the construction, $h_0(t)$ is called the scaling function and $h_1(t)$ the mother wavelet. There are two basic operations involved in this set of Haar functions: (1) translation and (2) dilation [25]. Starting from the mother wavelet, $h_1(t)$, compression and translation are performed to obtain $h_2(t)$ and $h_3(t)$ as shown in Fig. 1. Any function $y(t)$ which is square integrable in the interval $0 \leq t \leq 1$, that is

$$\int_0^1 y^2(t)\,dt < \infty \tag{2.5}$$

can be expanded into Haar series by

$$y(t) = c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \cdots, \tag{2.6}$$

where $c_j = \int_0^1 y(t) h_j(t)\,dt$.

   In usual, the series expansion of (2.6) contains infinite terms for a general smooth function $y(t)$. However, if $y(t)$ is approximated as piecewise constantduring each subinterval, (2.6) will be terminated at finite terms, i.e.

$$y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) \tag{2.7}$$

The continuous curve of (2.7) can be written into the discrete form by

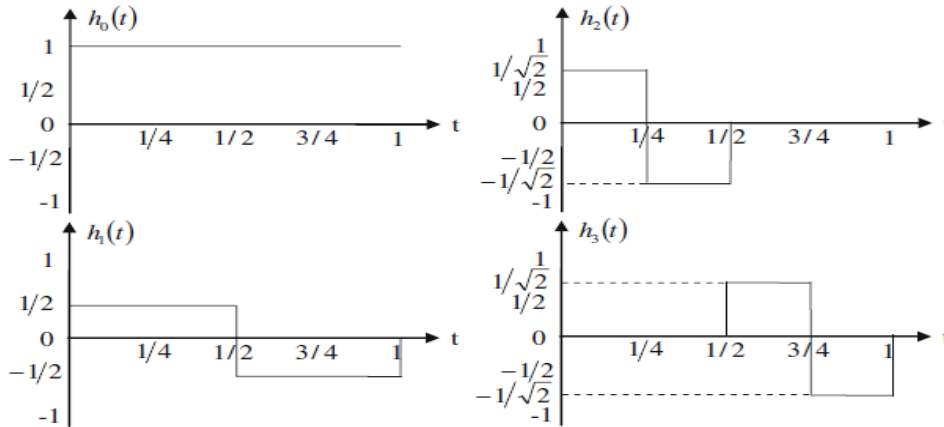$$y(t) \approx c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \cdots + c_{m-1} h_{m-1}(t), \tag{2.8}$$

Fig 1: Haar wavelets functions with $m = 4$.

   where m is the dimension and usually $m = 2^r$, is a positive integer. $\vec{y}^T = [y_0, y_1, \ldots, y_{m-1}]$is the discrete form of the continuous function $y(t)$, the discrete values yi are obtained by sampling the continuous curve $y(t)$ at a space $\dfrac{1}{m}$. Similarly,

$\vec{h_0}^T = [h_{0,0}, h_{0,1}, h_{0,2}, \ldots, h_{0,m-1}], \vec{h_1}^T = [h_{1,0}, h_{1,1}, h_{1,2}, \ldots, h_{1,m-1}], \ldots, \vec{h}_{m-1}^T = [h_{m-1,0}, h_{m-1,1}, h_{m-1,2}, \ldots, h_{m-1,m-1}]$ of the Haar wavelet bases; the discrete values are taken form the continuous curves $h_0(t), h_1(t), \ldots, h_{m-1}(t)$, respectivly

$$H = \begin{pmatrix} \vec{h}_o(t) \\ \vec{h}_1(t) \\ \cdots \\ \vec{h}_{m-1}(t) \end{pmatrix} = \begin{pmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \cdots & . & \cdots & \cdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{pmatrix} \qquad (2.9)$$

Eq.(2.8) is then expressed $\vec{y}^T = \vec{c}^T.H$, where $\vec{c} = [c_0, c_1, \ldots, c_{m-1}]^T$ is called the coefficient vector of $\vec{y}$ and it can be calculated from $\vec{c}^T = \vec{y}^T.H^{-1}$. Similarly, a two-dimensional function $y(x,t)$ which is square integrable in the interval $0 < x < 1$ and $0 < t < 1$ can be expanded into Haar series by

$$y(x,t) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t) \qquad (2.10)$$

where

$$c_{ij} = \int_0^1 y(x,t) h_i(x) \, dx \int_0^1 y(x,t) h_i(t) \, dt . \qquad (2.11)$$

Thus, Eq. (2.10) can be written into the discrete form by

$$Y(x,t) = H^T(x) \cdot C \cdot H(t) \qquad (2.12)$$

where

$$C = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1} \\ \cdots & . & \cdots & \cdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1} \end{pmatrix} \qquad (2.13)$$

is the coefficient matrix of Y , and it can be calculated by

$$C = H \cdot Y \cdot H^T . \qquad (2.14)$$

For deriving the operational matrix of Haar wavelets, we let $\Phi = H$ in Eq. (2.3), and obtain

$$F_H = H \cdot F_B \cdot H^T, \qquad (2.15)$$

where F_H is the operational matrix for integration of H. For example, the operational matrix of the Haar wavelet in the case of m = 4 is given by

$$F_{H_4} = H_4 \cdot F_{B_4} \cdot H^T$$

$$
= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} .5 & -.25 & -.0884 & -.0884 \\ .25 & 0 & -.0884 & .0884 \\ .0884 & .0884 & 0 & 0 \\ .0884 & -.0884 & 0 & 0 \end{pmatrix} .
$$

For any $m = 2^r$ where r is a positive integer, we can establish the corresponding operational matrix accordingly.

3. Numerical evaluation of fractional calculus

Fractional calculus is a generalization of integration and differentiation to non-integer order. In this section, we use the operational matrix of orthogonal functions to express the fractional derivatives [26]. It consists of the following three steps:

Step 1: Sample the continuous function $f(t)$ into the discrete vector form $\vec{f} = [f_0, f_1, ..., f_{m-1}]^T$, where $\vec{f}$ is a column vector expression.

Step 2: Transfer the vector $\vec{f}$ into the Haar wavelet domain by using the Haar wavelets transform

$$\vec{f}^T = \vec{c}^T \cdot H , \tag{3.1}$$

where c^T is the coeffcient vector and H is the Haar wavelet matrix.

Step 3: To find the numerical solution of $D_t^\alpha f = \dfrac{d^\alpha f}{dt^\alpha}$, where α is a real number, we use

$$D_t^\alpha f = D_t^\alpha (\vec{c} \cdot H) = \vec{c} \, D_t^\alpha (H) \tag{3.2}$$

From the definition of the operation matrix, it yields:

$$D_t^\alpha H = F_H^{-\alpha} \cdot H \tag{3.3}$$

where F_H is the operational matrix for integration of Haar wavelets. Substituting (3.3) into (3.2) yields:

$$D_t^\alpha f = D_t^\alpha (\vec{c}.H) = \vec{c} \, F_H^{-\alpha} \cdot H \tag{3.4}$$

where $F_H^{-\alpha}$ is Haar wavelet operational matrix with fractional order and it can be derived from the following equation

$$F_H^\alpha = H \cdot F_B^\alpha \cdot H^T , \tag{3.5}$$

where F_B^α is the operational matrix of the block pulse function for integration with the order α, i.e.

$$
F_B^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & & \xi_{m-2} \\ & 0 & 1 & \ddots & & \xi_{m-3} \\ \vdots & & & \ddots & & \vdots \\ & & & & \ddots & \xi_1 \\ 0 & & \cdots & & 0 & 1 \end{bmatrix} \tag{3.6}
$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

4. Analysis of the method

The numerical method for the linear fractional partial differential equation is illustrated in this section. This is based on the operational matrices of Haar wavelets.

Obviously, (3.4) can be extended to the case of a function with two variables, the integration order α of $Y(x,t) = H^T \cdot C \cdot H(t)$ with respect to variable t yields

$$I_t^\alpha Y(x,t) = I_t^\alpha (H^T(x) \cdot C \cdot H(t))$$
$$= H^T(x) \cdot C \cdot I_t^\alpha H(t) = H^T(x) \cdot C \cdot F_H^\alpha \cdot H(t)$$

or

$$I_t^\alpha Y = H^T \cdot C \cdot F_H^\alpha \cdot H \tag{4.1}$$

Similarly, the fractional integration order α of $Y(x,t)$ with respect to variable x can be expressed as

$$I_x^\beta Y(x,t) = I_x^\beta (H^T(x) \cdot C \cdot H(t))$$
$$= \left[I_x^\beta H(x) \right]^T \cdot C \cdot H(t) = F_H^\beta \cdot H^T(x) \cdot C \cdot H(t) = H^T(x) \cdot (F_H^\beta)^T \cdot C \cdot H(t)$$

or

$$I_x^\beta Y = H^T \cdot (F_H^\beta)^T \cdot C \cdot H \tag{4.2}$$

In general, performing the double integration to the function $Y(x,t)$ with fractional order α to variable t and fractional order β to variable x , we obtain

$$I_t^\alpha I_x^\beta Y = H^T \cdot (F_H^\beta)^T \cdot C \cdot F_H^\alpha \cdot H . \tag{4.3}$$

Similarly, we have

$$I_x^\beta I_t^\alpha Y = H^T \cdot (F_H^\beta)^T \cdot C \cdot F_H^\alpha \cdot H . \tag{4.4}$$

5. Applications and results

Example 1. We consider the following fractional Heat equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\pi^2} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{5.1}$$

along with the following initial and boundary conditions

$$u(0,t)=0, \quad \frac{\partial u}{\partial x}(0,t)= \pi E_\alpha(-t^\alpha), \quad t>0 \tag{5.2}$$

$$u(0,t)= sin\pi x, \quad x>0 \tag{5.3}$$

The exact solution of (4.1)-(4.3) is $u(x,t) = sin\pi x \, E_\alpha(-t^\alpha)$.

Now, we consider (4.1) with the following conditions

$$u(0,t)=0, \quad \frac{\partial u}{\partial x}(0,t)=x, \quad t>0 \tag{5.4}$$

$$u(0,t)=\pi x, \quad x>0 \tag{5.5}$$

Applying the double Laplace transform to both sides of (5.1), we have

$$s_2^{\alpha}\left[L_t\,L_x\,u(x,t)-s_2^{-1}\frac{\pi}{s_1^{-1}}\right]=\frac{1}{\pi^2}\left[s_1^{\alpha}\,L_t\,L_x\,u(x,t)-\frac{\pi}{s_2}\right].$$

Hence,

$$(s_2^{\alpha}-\frac{s_1^2}{\pi})L_t\,L_x\,u(x,t)=s_2^{\alpha-1}\frac{\pi}{s_1^2}-\frac{1}{\pi\,s_2}=\frac{\pi^2\,s_2^{\alpha}-s_1^2}{\pi\,s_1^2\,s_2}.$$

Thus, we have

$$L_t\,L_x\,u(x,t)=\frac{\pi}{s_1^2}\cdot\frac{1}{s_2}.$$

Consequently, $u(x,t)=\pi x$.

   Now, we apply the procedure in previous section for (5.1),(5.4) and (5.5). Operating the Riemann-Liouville operator I_t^{α} on both sides of Eq.(5.1)

$$u(x,t)-u(x,0)=\frac{1}{\pi^2\,\Gamma(\alpha)}\int_0^t(s-t)^{\alpha-1}u''_x\,(x,s)\,ds=\frac{1}{\pi^2}I_t^{\alpha}\,u''_x(x,t)\,. \tag{5.6}$$

Then, integrating (5.6) with respect to x two times, we get

$$\int_0^x\int_0^x u(x,t)\,dx\,dx-\pi\int_0^x\int_0^x\int_0^x dx\,dx\,dx=\frac{1}{\pi^2}\left(I_t^{\alpha}u(x,t)-\pi\int_0^x I_t^{\alpha}(1)\,dx\right) \tag{5.7}$$

Let $u(x,t)=H^T(x).C.H(t)$. Substituting (4.1), (4.2) and (4.4) into (5.7), we obtain

$$H^T(x)\cdot(F_H^2)^T\cdot C\cdot H(t)-\pi\,H^T(x)\cdot(F_H^3)^T\cdot J\cdot H(t)$$
$$=\frac{1}{\pi^2}H^T(x)\cdot C\cdot(F_H^{\alpha})^T\cdot H(t)-\frac{1}{\pi}H^T(x)\cdot(F_H^1)^T\cdot J\cdot F_H^{\alpha}\cdot H(t) \tag{5.8}$$

where

$$J=H.\begin{pmatrix}1&1&\cdots&1\\1&1&\cdots&1\\\vdots&.&\vdots&\vdots\\1&1&\cdots&1\end{pmatrix}_{m\times m}\cdot H^T\,. \tag{5.9}$$

By multiplying $H^T(x)$ to the right side and H(t) to the left side of each term in (5.8), it gives

$$(F_H^2)^T\cdot C-\pi(F_H^3)^T\cdot J=\frac{1}{\pi^2}\cdot C\cdot F_H^{\alpha}-\frac{1}{\pi}(F_H^1)^T\cdot J\cdot F_H^{\alpha} \tag{5.10}$$

If the Haar wavelets with order m is used, (5.10) can be simplifed as:

$$(F_{H_m}^2)^T\cdot C_{m\times m}-\pi(F_{H_m}^3)^T\cdot J_m=\frac{1}{\pi^2}\,C_{m\times m}\cdot F_{H_m}^{\alpha}-\frac{1}{\pi}(F_{H_m}^1)^T\cdot J_m\cdot F_{H_m}^{\alpha} \tag{5.11}$$

Solving the above equation to $m=8$ and $\alpha=\dfrac{1}{2}$, the matrix form of $u(x,t)$ is given by

$$
\begin{bmatrix}
109.1780 & 116.1585 & 88.4466 & 75.2536 & -24.6560 & -156.6537 & -342.2363 & -586.5817 \\
860.1673 & 822.8567 & 970.9540 & 1041.566 & 1575.925 & 2281.514 & 2371.886 & 4572.098 \\
1077.383 & 1136.713 & 901.1135 & 789.2779 & -58.8275 & -1180.506 & -2762.631 & -4857.286 \\
3851.943 & 3375.387 & 5266.210 & 6171.852 & 13009.15 & 22023.91 & 34617.70 & 51016.62 \\
-10971.47 & -8817.750 & -17360.79 & -21462.56 & -52400.09 & -93148.34 & -1.499\ 10^5 & -2.234062^5 \\
50170.69 & 42248.90 & 73663.42 & 88787.45 & 2.0271383^5 & 3.526208^5 & 5.6076830^5 & 8.2889910^5 \\
-1.764110^5 & -1.46910^5 & -2.638\ 10^5 & -3.20310^5 & -7.45010^5 & -1.30310^6 & -2.07610^6 & -3.06710^6 \\
6.7301010^7 & 5.641410^5 & 9.956010^5 & 1.204410^6 & 2.773410^6 & 4.834110^6 & 7.678210^6 & 1.130210^7
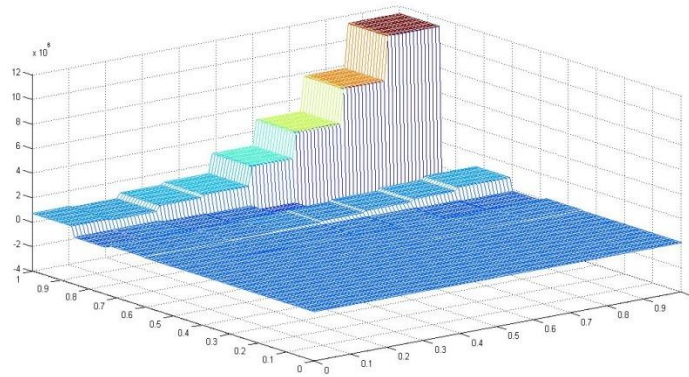\end{bmatrix}
$$

Figure 2: The approximate solution of (4.1)-(4.3) in the case $\alpha=1/2$.

Example 2. Consider the following fractional Wave equation

$$
\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = k^2 \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}, \quad 1 < \alpha < 2, \, 1 < \beta < 2 \tag{5.12}
$$

along with the following initial and boundary conditions

$$
u(0,t)=0, \quad \frac{\partial u}{\partial x}(0,t)=b_2, \quad t>0 \tag{5.13}
$$

$$
u(x,0)=0, \quad \frac{\partial u}{\partial x}(x,0)=b_1, \quad t>0 \tag{5.14}
$$

where b_1 and b_2 are constants.

Operating the Riemann-Liouville operator I_t^{α} on both sides of Eq.(5.12) and with boundary conditions (5.14) we have

$$
u(x,t) - \int_0^x dx - b_1 \int_0^t dt = k^2 I_t^{\alpha} \left(\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} \right). \tag{5.15}
$$

Operating the Riemann-Liouville operator I_x^β on both sides of Eq.(5.15) and with boundary conditions (5.13) we get

$$I_x^\beta u(x,t) - I_x^{\beta+1}(1) - b_1 I_x^\beta I_t^1(1) = k^2 \left(I_t^\alpha u(x,t) - b_2 I_t^\alpha I_x^1(1) \right). \tag{5.16}$$

Let $u(x,t) = H^T(x) \cdot C \cdot H(t)$. Substituting (4.1)-(4.4) into (5.16), we obtain

$$H^T(x) \cdot (F_H^\beta)^T \cdot H(t) - H^T(x) \cdot (F_H^{\beta+1})^T \cdot J \cdot H(t) - b_1 H^T(x) \cdot (F_H^\beta)^T \cdot J \cdot F_H^1 \cdot H(t)$$
$$= k^2 H^T(x) \cdot C \cdot F_H^\alpha \cdot H(t) - k^2 b_2 H^T(x) \cdot (F_H^1)^T \cdot J \cdot F_H^\alpha \cdot H(t),$$

where J is same as (5.9) .

By multiplying $H^T(x)$ to the right side and $H(t)$ to the left side of each term in the above equation, for Haar wavelets with order m, we have

$$(F_{H_m}^\beta)^T \cdot C_{m \times m} - (F_{H_m}^{\beta+1})^T \cdot J_m - b_1 (F_{H_m}^\beta)^T \cdot J_m$$
$$= k^2 C_{m \times m} \cdot F_{H_m}^\alpha - K^2 b_2 (F_{H_m}^1)^T \cdot J_m \cdot F_{H_m}^\alpha . \tag{5.17}$$

Example 3. Consider the following fractional Diffusion-Wave equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 1 < \alpha \le 2, \tag{5.18}$$

subjected to the initial conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = b_2, \quad t > 0 \tag{5.19}$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial x}(x,0) = b_1, \quad t > 0, \tag{5.20}$$

where b_1 and b_2 are constants.

Operating the Riemann-Liouville operator I_t^α on both sides of Eq.(5.18) and with boundary conditions (5.20) we have

$$u(x,t) - \int_0^x dx - b_1 \int_0^t dt = I_t^\alpha u_x''(x,t). \tag{5.21}$$

Then, integrate (5.21) with respect to x two times, we get

$$u(x,t) - \int_0^x dx - \int_0^x \int_0^x \int_0^x dx\, dx\, dx \ - b_1 \int_0^x \int_0^x \int_0^t dt\, dx\, dx$$
$$= I_t^\alpha u_x(x,t) - I_t^\alpha \int_0^x b_2\, dx. \tag{5.22}$$

Let $u(x,t) = H^T(x) \cdot C \cdot H(t)$. Substituting (4.1)-(4.4) into (5.22), we obtain
$$H^T(x) \cdot (F_H^2)^T \cdot C \cdot H(t) - H^T(x) \cdot (F_H^3)^T \cdot J \cdot H(t) - b_1 H^T(x) \cdot (F_H^2)^T \cdot J \cdot F_H^1 \cdot H(t)$$
$$= H^T(x) \cdot C \cdot F_H^\alpha \cdot H(t) - b_2 H^T(x) \cdot (F_H^1)^T \cdot J \cdot F_H^\alpha \cdot H(t).$$

By multiplying $H^T(x)$ to the right side and $H(t)$ to the left side of each term in the above equation, for Haar wavelets with order m, we have

$$(F_{H_m}^2)^T \cdot C_{m \times m} - (F_{H_m}^3)^T \cdot J_m - b_1 (F_{H_m}^2)^T \cdot J_m \cdot F_{H_m}^1$$
$$= C_{m \times m} \cdot F_{H_m}^\alpha - b_2 (F_{H_m}^1)^T \cdot J_m \cdot F_{H_m}^\alpha$$

(5.23)

In the case of $m = 8, \alpha = \dfrac{1}{2}, b_1 = 1$ and $b_2 = 2$, we get the numerical solution by solving the equation as below

$$(F_{H_8}^2)^T \cdot C_{8 \times 8} - (F_{H_8}^3)^T \cdot J_8 - b_1 (F_{H_8}^2)^T \cdot J_8 \cdot F_{H_8}^1 = C_{8 \times 8} \cdot F_{H_8}^\alpha - b_2 (F_{H_8}^1)^T \cdot J_8 \cdot F_{H_8}^\alpha \qquad (5.24)$$
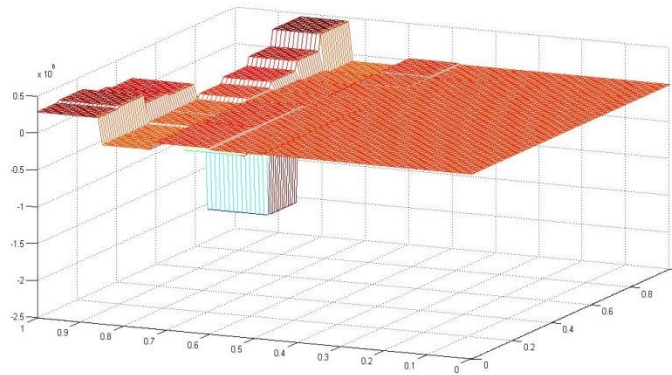
See Figure3.

Figure 3: The approximate solution of (4.1)-(4.3) in the case $\alpha = 1/2, b_1 = 1, b_2 = 2$.

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