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Fixed Point Theorems for Weakly Compatible Maps Under E.A. Property in Fuzzy 2-Metric Spaces

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Abstract

In this paper, we introduce E.A. property on fuzzy 2-metric spaces and prove common fixed point theorem for a pair of weakly compatible maps under E.A. property on fuzzy 2-metric spaces.

Keywords: Weakly compatible maps, E.A. property, fuzzy 2-metric space.

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by L.A. zadeh [17] in 1965. After this fuzzy set theory was further developed and a series of research were done by several mathematicians. In the sequel the concept of fuzzy metric space was introduced by Kramosil and Michalek [13] in 1975. M . Grabies[8] proved the contraction principle in fuzzy metric spaces with the help of t-norms in 1988. Moreover, A. George and P. Veeramani [7] modified the notion of fuzzy metric spaces with the help of t-norm in 1994. Aamri and El. Moutawakil [1] generalized the concepts of non-compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition.

Gähler investigated 2-metric spaces in a series of his papers [4],[5],[6].

It is to be remarked that Sharma, Sharma and Iseki [10] studied for the first time contraction type mapping in 2-metric space. S.H. Cho [2] proved a common fixed point theorem for four mappings in fuzzy metric space and S. Sharma [15] proved a common fixed point theorem for three mappings in fuzzy 2-metric space.
In this paper we prove common fixed point theorems for weakly compatible maps in fuzzy 2-metric space by using the concept of (E.A) property.

Definition 1.1: A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval [0,1] such that for all a, b, c, d∈[0,1] . The following conditions are satisfied:

1) $\boldsymbol{a * 1 = a}$;
2) $\boldsymbol{a * b = b * a}$;
3) $\mathbf{a * b \leq c * d}$ **whe**never a ≤ c and b ≤ d
4) $\boldsymbol{a * (b * c) = (a * b) * c}$.

Definition 1.2. The 3-tuple $(\boldsymbol{X}, \boldsymbol{M}, *)$ is called a fuzzy metrice spase, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $\boldsymbol{X^2} \times [\mathbf{0}, \infty]$ satisfying the following condition:

for all x, y, z ∈ X and s, t>0

C'-1 $\boldsymbol{M(x, y, 0) = 0}$

C'-2 $\boldsymbol{M(x, y, t) = 1}$, **for all** $\boldsymbol{t > 0}$, $\boldsymbol{if\ and\ only\ if\ x = y}$

C'-3 $\boldsymbol{M(x, y, t) = M(y, x, t)}$

C'-4 $\boldsymbol{M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)}$

C'-5 $\boldsymbol{M(x, y, .)}: [\mathbf{0}, \infty) \to [\mathbf{0}, \mathbf{1}]$ **is left contininuos**,

C'-6 $\lim_{t \to \infty} \boldsymbol{M(x, y, t) = 1}$

Example 1.3. let (X, d) be a metric space. Define $\boldsymbol{a * b = ab}$ (or $\boldsymbol{a * b} = \min\{\boldsymbol{a, b}\}$) and for all x,y∈X and t>0, $\boldsymbol{M(x, y, t)} = \frac{t}{t + d(x, y)}$. Then $(\boldsymbol{X}, \boldsymbol{M}, *)$ is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 1.4. A binary operation $*: [\mathbf{0}, \mathbf{1}] \times [\mathbf{0}, \mathbf{1}] \times [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ is called a continuous t-norm if $([\mathbf{0}, \mathbf{1}], *)$ is an abelain topological monoid with unit 1such that $\boldsymbol{a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2}$ whenever

$\boldsymbol{a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2}$ **for all** $\boldsymbol{a_1, a_2, b_1, b_2}$ **and** $\boldsymbol{c_1, c_2}$ $\boldsymbol{in}$ $[\mathbf{0}, \mathbf{1}]$.

Definition 1.5. The 3-tuple $(\boldsymbol{X}, \boldsymbol{M}, *)$ is called a fuzzy 2-metric space if X is an arbitarary set, $*$ is a continuous t-norm and M is a fuzzy set in $\boldsymbol{X^3} \times [\mathbf{0}, \infty]$ satisfying the following conditions, for all *x, y, z, u ∈X* and $t_1, t_2, t_3 > 0$.

C''-1 $\boldsymbol{M(x, y, z, 0) = 0}$,

C''-2 $M(x,y,z,t) = 1, t > 0$ and when at least two of the three point are equal,

C''-3 $M(x,y,z,t) = M\ (x,z,y,t) = M\ (y,z,x,t)$

C''-4 $M(x,y,z,t_1 + t_2 + t_3) \geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$

(This correspond to tetrahedron inequality in 2-metric space)

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

C''-5 $M(x,y,z,.)\colon [0,\infty) \to [0,1]$ is left continuous.

Example 1.6: let (X, d) be 2-metric space and denote $a * b = ab$ for all a,b∈[0,1].

For each $h, m, n \epsilon R^+$ and t>0, define $M(x,y,z,t) = \frac{ht^n}{ht^n + md(x,y,z)}$.

Then $(X, M, *)$ is an fuzzy 2-metric space.

Definition 1.7: A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M, *)$ is said to converge to x($in\ X$) if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all a∈X and t>0.

Definition 1.8: Let $(X, M, *)$ be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M\left(x_{n+p}, x_n, a, t\right) = 1$ for all $a \epsilon X,\ p > 0, t > 0$.

Definition 1.9: A fuzzy 2-metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 1.10: Let f and g mapping from a fuzzy 2-metric space $(X, M, *)$ into itself. A pair of map$\{f, g\}$is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, a, t) = 1$,whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some u∈X and for all t>0.

Definition 1.11: Let f and g mapping from a fuzzy 2-metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to be non-compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, a, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some u ∈X and for all t>0.

Definition 1.12: Let $(X, M, *)$ be a fuzzy 2-meric space. Suppose f and g be self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.

Definition 1.13: A pair of self mapping $\{f, g\}$ of a fuzzy 2-metric space (X, d) is said to be weakly compatible if they commute at the coincidence point i.e., If $fu = gu$ for some $u \epsilon X$, then $fgu = gfu$.

It is to see that two compatible maps are weakly compatible but converse is not true.

2.Main results

Definition 2.1: Let f and g be two self-maps of a 2-metric space $(X, M,*)$ then they are said to satisfy E.A property if there exists a sequence $\{x_n\}$ in X such that:

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, for\ some\ t\epsilon X.$$

Now in a similar mode we state E.A. property in fuzzy 2-metric spaces as follow:

Definition 2.2: A pair of self-mapping $\{f, g\}$ of a fuzzy 2-metric spaces $(X, M,*)$ is said to satisfy E.A property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(fx_n, gx_n, a, t) = 1\ for\ some\ t\epsilon X.$$

Example 2.3: Let $X = [2, \infty)$. Define $f, g: X \to X$ by $gx = x + 1$ and $fx = 2x + 1$, for all $x\epsilon X$. suppose that the E.A. property holds. Then, there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some zϵX.

Therefore, $\lim_{n\to\infty} x_n = z - 1$ and $\lim_{n\to\infty} x_n = \frac{z-1}{2}$. Thus, z=1, which is a contraction, since 1 is not contain in X. Hence f and g do not satisfy E.A property.

Notice that weakly compatible and E.A. property are independent to each other.

Example 2.4: Let X=[0,1] and $d(x, y, z) = |x(y - z) + y(z - x) + z(x - y)|$ be a 2-metric space on X. Define $M(x, y, z, t) = (\frac{t}{t+d(x,y,z)})$ for all $x, y, z\epsilon X$ and for all t>0 and also define.

$$fx = \begin{cases} 1 - x & if\ x\epsilon[0, \frac{1}{2}] \\ 0 & if\ x\epsilon(\frac{1}{2}, 1] \end{cases} \qquad gx = \begin{cases} \frac{1}{2} & if\ x\epsilon[0, \frac{1}{2}] \\ \frac{3}{4} & if\ x\epsilon(\frac{1}{2}, 1] \end{cases}$$

Consider the sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}$, $n \geq 0$, we have $\lim_{n\to\infty} f(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = \lim_{n\to\infty} g(\frac{1}{2} - \frac{1}{n})$. Thus, the pair (f, g) satisfies E.A. property.

Further, f and g are weakly compatible since $x = \frac{1}{2}$ is their unique coincidence point and $fg\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = gf\left(\frac{1}{2}\right)$. We further observe that

$\lim_{n\to\infty} d(fg(\frac{1}{2} - \frac{1}{n}), gf(\frac{1}{2} - \frac{1}{n}), z) \neq 0$, showing that $\lim_{n\to\infty} M(fgx_n, gfx_n, z, t) \neq 1$, therefore, the pair (f, g) is non-compatible.

Example 2.5: Let $X = R^+$ and $d(x, y, z) = |x(y - z) + y(z - x) + z(x - y)|$ be a 2-metrice space on X. Define $M(x, y, z, t) = \frac{t}{t+d(x,y,z)}$ for all $x, y, z\epsilon X$ and for all t>0 and also define $f, g: X \to X$ by

$fx = 0$ if $0 < x \leq 1$ and $fx = 1, if\ x > 1\ or\ x = 0$; and $gx = [x]$, (the greatest integer that is less than or equal to x,) for all $x \epsilon X$. Consider a sequence $\{x_n\} = \left\{1 - \frac{1}{n}\right\}, n \geq 2$ in $(0, 1)$, we have $\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n$. Thus the pair (f, g) satisfies E.A. property. However, f and g are not weakly compatible as as each $u_1 \epsilon (0, 1)$ and $u_2 \epsilon (1, 2)$ are coincidence point of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2, \ldots$ but none of these point are coincidence points of f and g. Further, we note that pair (f, g) is non compatible. Thus we can conclude that, E.A. property dose not imply weak compatibility. Here, we notice that weakly compatible and E.A. property are independent to each other.

Lemma 2.6: Let $(X, M, *)$ be a fuzzy 2-metric space. If there exists $k \epsilon (0, 1)$ such that $M(x, y, z, kt) \geq M(x, y, z, t)$ **for all** $x, y, z \epsilon X$ **with** $z \neq x, z \neq y$ **and** $t > 0$, **then** $x = y$.

Proof. Since $(x, y, z, t) \geq M(x, y, z, kt) \geq M(x, y, z, t)$, then $M(x, y, z, .)$ is constant for all x, y, z,. Since $\lim_{t\to\infty} M(x, y, z, t) = 1, M(x, y, z, t) = 1$ for all $t>0$. Hence, $x = y$ because $x \neq z$ and $y \neq z$.

Theorem 2.7. Let f and g be weakly compatible self maps of a fuzzy2-metric spaces $(X, M, *)$, satisfying The following:

(a-1) $M(x, y, z, t) > 0$ $for\ all\ x, y\ in\ X\ and\ t > 0$

(a-2) $M(fx, fy, z, t) \geq r\big(M(gx, gy, z, t)\big), for\ all\ x, y\ in\ X\ and\ t > 0$, Where $r: [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1, r(0) = 0$ and $r(1) = 1$

(a-3) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $\{y_n\} \to y$, and $t > 0$, then $M(x_n, y_n, z, t) \to M(x, y, z, t)$.

(a-4) f and g satisfy the E.A. propery,

(a-5) $g(X)$ is a closed subspace of X .

Then f and g have a unique common fixed point in X.

Proof: Since f and g satisfy the E.A. property therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \epsilon X$. As $g(X)$ is a closed subspace of X, therefore every convergent sequence of points of $g(X)$ has a limit point in $g(X)$.

Therefore, $u = lim_{n\to\infty} gx_n = ga = lim_{n\to\infty} fx_n$ for some $a \epsilon X$. This implies $u = ga\ \epsilon g(X)$.

Now we show that $fa = ga$. From (a-2), we have, $M(fa, fx_n, z, t) \geq r\big(M(ga, gx_n, z, t)\big)$.

Proceeding limit as $n \to \infty$, we have

$M(fa, u, z, t) \geq r\big(M(u, u, z, t)\big) = r(1) = 1$, this implies that $u = ga = fa$.

Thus a is the coincidence point of f and g.

Since f and g are weakly compatible, therefore, $fu = fga = gfa = gu$.

Now we show that $fu = u$. From (a-2), we have $M(fu, fa, z, t) \geq r\big(M(gu, ga, z, t)\big)$, which in turns implies that $fu = u$. Hence u is the unique common fixed point of f and g.

Uniqueness follows easily from (a-2).

Consider the mapping $\Phi : [0,1]^5 \to [0,1]$, which is upper semicontinuous, non-decreasing in each coordinate variable a and such that

$$\phi(1, t, 1, t, 1) \geq t, \Phi(1, 1, t, t, 1) \geq t, \phi(1, 1, 1, t, t) \geq t (t \epsilon [0, 1]).$$

Now we prove a common fixed point theorem for pair of mappings using control function under E.A. property provided maps are weakly compatible.

Theorem 2.8. Let A , B , S and T be self maps of a fuzzy 2-metric spaces $(X, M, *)$ satisfying the following condition:

1. A(X) $\subset T(X)$ *and* $B(X) \subset S(X)$,
2. M(Ax,Bx,z,kt)≥ϕ(M(Sx,Ty,z,t),M(Ax,Sx,z,t),M(By,Ty,z,t),M(Sx,By,z,t),M(Ax,Ty,z,t)), for all x, y, z in X and $t > 0$, where $k \epsilon (0, 1)$,
3. pairs (A,S) or (B,T) satisfy E.A. property,
4. pairs (A,S) or (B,T) are weakly compatible.

If the range of one of A, B, S and T is a closed subset of X, then A, B, S and T have a Unique common fixed point in X.

Proof: suppose that (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = u$ for some $u \epsilon X$. Since B(X) ⊂ S(X), therefore, there exists a sequence $\{y_n\} \epsilon X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = u$. Hence $\lim_{n\to\infty} Sy_n = u$.

Now we shall show that $\lim_{n\to\infty} Ay_n = u$. suppose that $\lim_{n\to\infty} Ay_n = l$. Therefore, from (2) we have

$M(Ay_n,Bx_n,z,kt) \geq \phi(M(Sy_n,Tx_n,z,t),M(Ay_n,Sy_n,z,t),M(Bx_n,Tx_n,z,t),M(Sy_n,Bx_n,z,t),M(Ay_n,Tx_n,z,t))$.

Proceeding limit as $n \to \infty$, we have;

$M(l, u, z, kt) \geq \phi\big(1, M(l, u, z, t), 1, 1, M(l, u, z, kt)\big) \geq M(l, u, z, t)$, by lemma 1.19,we have l = u.

Therefore, we have $\lim_{n\to\infty} Ay_n = u$. Suppose that S(X) is a closed subspace of X. Then $u = Sv$ for some $v \epsilon X$. Subsequently, we have ;

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = u = Sv.$$

Now, we shall show that $Av = u$.

From (2) we have

$$M(Av,Bx_n,z,kt)\geq\phi(M(Sv,Tx_n,z,t),M(Av,Sv,z,t),M(Bx_n,Tx_n,z,t),M(Sv,Bx_n,z,t),M(Av,Tx_n,z,t)).$$

Letting limit as $n\to\infty$,

$M(Av,u,z,kt)\geq\phi\big(1,M(Av,u,z,t),1,1,M(Av,u,z,t)\big)\geq M(Av,u,z,t)$, by lemma 1.19, we have

$Av=Sv=u$. Since A(X) ⊂ T(X), so there exists $w\epsilon X$ such that $u=Av=Tw$.

Now, we claim that $u=Bw$. From (2) we have

$$M(Av,Bw,z,kt)\geq\phi\big(M(Sv,Tw,z,t),M(Av,Sv,z,t),M(Bw,Tw,z,t),M(Sv,Bw,z,t),M(Av,Tw,z,t)\big)$$

Or $M(u,Bw,z,kt)\geq\phi(1,1,M(Bw,u,z,t),M(u,Bw,z,t),1)\geq M(u,Bw,z,t)$, by lemma 1.19, we have $u=Bw$.

Thus we have $Av=Sv=Tw=Bw=u$.

Since the pair (A , S) is weak compatible, therefore, $ASv=SAv$ $i.e$, $Au=Su$.

Now we show that $Au=u$. Since

$$M(Au,Bw,z,kt)\geq\phi\big(M(su,Tw,z,t),M(Au,Su,z,t),M(Bw,Tw,z,t),M(Su,Bw,z,t),M(Au,Tw,z,t)\big)$$

Or $M(Au,u,z,kt)\geq\phi\big(M(Au,u,z,t),1,1,M(Au,u,z,t),M(Au,u,z,t)\big)\geq M(Au,u,z,t)$, by lemma 1.19, we have $Au=Su=u$.

The weak compatibility of B and T implies that $BTw=TBw, i.e, Bu=Tu$.

Now we shall further show that u is the common fixed point of B.

From (2) , one obtain

$$M(Au,Bu,z,kt)\geq\phi\big(M(su,Tu,z,t),M(Au,Su,z,t),M(Bu,Tu,z,t),M(Su,Bu,z,t),M(Au,Tu,z,t)\big).$$

or $M(Au,Bu,z,kt)\geq\phi(M(Au,Bu,z,t),1,1,M(Au,Bu,z,t),1)$, by lemma 1.19, $Bu=u$.

Hence $Au=Bu=Su=Tu=u$ and u is a common fixed point of A, B, S and T□

Corollary 2.9: let A,B,S and T be self maps of a fuzzy 2-metric space $(X,M,*)$ with continuous t-norm satisfying (1),(3),(4) and the following:

(5)

$$M(Ax,By,z,t)\geq\min\{M(Sx,Ty,z,t),M(Ax,Sx,z,t),M(By,Ty,z,t),M(Sx,By,z,t),M(Ax,Ty,z,t)\}$$

Holds, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\boldsymbol{X}$ and t>0.

If the range of one of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}$ and $\boldsymbol{T}$ is closed subset of $\boldsymbol{X}$, then $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}$ and $\boldsymbol{T}$ have a unique common fixed point in X.

Proof: Take in the above Theorem $\boldsymbol{\phi}(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \boldsymbol{x_4}, \boldsymbol{x_5}) = \mathbf{min}\, \{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \boldsymbol{x_4}, \boldsymbol{x_5}\}$□

Next we consider a function $\boldsymbol{\psi}: [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ satisfying the conditions

$(*)$ $\boldsymbol{\psi}$ ***if continuous and nondecreasing on*** $[\mathbf{0}, \mathbf{1}]$, and $\boldsymbol{\psi}(\boldsymbol{t}) > \boldsymbol{t}$ ***for all t in*** $(\mathbf{0}, \mathbf{1})$.

We note that $\psi(1)=1$ and $\psi(t)\geq t$ for all t in [0,1], that is, $\boldsymbol{\psi}(\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})) \geq \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t})$ holds for every t>0 and for all $\boldsymbol{x}, \boldsymbol{y}$ in X.

Theorem 2.10: let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}$ ***and*** $\boldsymbol{T}$ be self maps of a fuzzy 2-metric space $(\boldsymbol{X}, \boldsymbol{M}, *)$ with continuous t-norm * satisfying (1),(3),(4) and the following:

(6) $\boldsymbol{M}(\boldsymbol{Ax}, \boldsymbol{By}, \boldsymbol{z}, \boldsymbol{t}) \geq$

$\boldsymbol{\psi}(\mathbf{min}\{\boldsymbol{M}(\boldsymbol{Sx}, \boldsymbol{Ty}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Ax}, \boldsymbol{Sx}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{By}, \boldsymbol{Ty}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Sx}, \boldsymbol{By}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Ax}, \boldsymbol{Ty}, \boldsymbol{z}, \boldsymbol{t})\})$

with $\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) > \boldsymbol{0}$ ***for all x, y, z in X and t*** $> \boldsymbol{0}$.

If the range of one of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}$ ***and*** $\boldsymbol{T}$ is a closed subset of X, then $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}$ ***and*** $\boldsymbol{T}$ have a unique common fixed point in X.

Proof: Suppose that $(\boldsymbol{A}, \boldsymbol{B})$ satisfies the E.A. property. Then there exists a sequence $\{\boldsymbol{x_n}\}$ in X such that $\mathbf{lim}_{n\to\infty} \boldsymbol{Bx_n} = \mathbf{lim}_n \boldsymbol{Tx}_{n\to\infty} = \boldsymbol{u}$ ***for some*** $\boldsymbol{u} \epsilon \boldsymbol{X}$.

Since $\boldsymbol{B}(\boldsymbol{X}) \subset \boldsymbol{S}(\boldsymbol{X})$ there exists a sequence $\{\boldsymbol{y_n}\} \epsilon \boldsymbol{X}$ such that $\boldsymbol{Bx_n} = \boldsymbol{Sy_n} = \boldsymbol{u}$. Hence $\mathbf{lim}_{n\to\infty} \boldsymbol{Sy_n} = \boldsymbol{u}$.

We shall show that $\mathbf{lim}_{n\to\infty} \boldsymbol{Ax_n} = \boldsymbol{u}$.

From(6) we have

$\boldsymbol{M}(\boldsymbol{Ay_n}, \boldsymbol{Bx_n}, \boldsymbol{z}, \boldsymbol{t}) \geq$
$\boldsymbol{\psi}(\mathbf{min}\{\boldsymbol{M}(\boldsymbol{Sy_n}, \boldsymbol{Tx_n}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Ay_n}, \boldsymbol{Sy_n}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Bx_n}, \boldsymbol{Tx_n}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Sy_n}, \boldsymbol{Bx_n}, \boldsymbol{z}, \boldsymbol{t}), \boldsymbol{M}(\boldsymbol{Ay_n}, \boldsymbol{Tx_n}, \boldsymbol{z}, \boldsymbol{t})\})$
.

Proceeding limit as $\boldsymbol{n} \to \infty$, one obtain, $\mathbf{lim}_{n\to\infty} \boldsymbol{Ay_n} = \boldsymbol{u}$.

Now, suppose that $\boldsymbol{S}(\boldsymbol{X})$ is a closed subspace of X. Then $\boldsymbol{u} = \boldsymbol{Sv}$ ***for some*** $\boldsymbol{v} \epsilon \boldsymbol{X}$. Subsequently we have;

$\mathbf{lim}_{n\to\infty} \boldsymbol{Ay_n} = \mathbf{lim}_{n\to\infty} \boldsymbol{Bx_n} = \mathbf{lim}_{n\to\infty} \boldsymbol{Tx_n} = \mathbf{lim}_{n\to\infty} \boldsymbol{Sy_n} = \boldsymbol{u} = \boldsymbol{Sv}$.

Now, we shall show that $\boldsymbol{Av} = \boldsymbol{Sv}$. From (6) we have

$$\boldsymbol{M(Av, Bx_n, z, t) \geq \psi(\min\{M(Sv, Tx_n, z, t), M(Av, Sv, z, t), M(Bx_n, Tx_n, z, t), M(Sv, Bx_n, z, t), M(Av, Tx_n, z, t)\})}.$$

Letting limit as $\boldsymbol{n} \to \infty$, we get

$$\boldsymbol{M(Av, u, z, t) \geq \psi(\min\{M(u, u, z, t), M(Av, u, z, t), M(u, u, z, t), M(u, u, z, t), M(Av, u, z, t)\})}$$

Using $(*)$, we have, $\boldsymbol{Av = Sv = u}$.

Since $\boldsymbol{AX \subset TX}$, so there exists $\boldsymbol{w \epsilon X}$ such that $\boldsymbol{u = Av = Tw}$.

Now, we claim that $\boldsymbol{u = Tw = Bw}$.

From (6) we have

$$\boldsymbol{M(Av, Bw, z, t) \geq \psi(\min\{M(Sv, Tw, z, t), M(Av, Sv, z, t), M(Bw, Tw, z, t), M(Sv, Bw, z, t), M(Av, Tw, z, t)\})},$$

Or

$$\boldsymbol{M(u, Bw, z, t) = \psi(\min\{M(u, u, z, t), M(u, u, z, t), M(Bw, u, z, t), M(u, Bw, z, t), M(u, u, z, t)\})},$$

Using $(*)$, we have, $\boldsymbol{u = Bw}$. Thus we have $\boldsymbol{Av = Sv = Tw = Bw = u}$. Since the pair $(\boldsymbol{A, S})$ is weak compatible which implies $\boldsymbol{ASv = SAv\ i.e, Au = Su}$.

From(6)

$$\boldsymbol{M(Au, Bw, z, t) \geq \psi(\min\{M(Su, Tw, z, t), M(Au, Su, z, t), M(Bw, Tw, z, t), M(Su, Bw, z, t), M(Au, Tw, z, t)\})}$$

By $(*)$, we have, $\boldsymbol{Au = Su = u}$.

The weak compatibility of B and T implies that $\boldsymbol{BTw = TBw, i.e., Bu = Tu}$.

Now we shall show that $\boldsymbol{u}$ is the common fixed point of $\boldsymbol{A, B, T\ and\ S}$.

Suppose that $\boldsymbol{Bu \neq u}$. then using (6) one obtain

$$\boldsymbol{M(Au, Bu, z, t) \geq \psi(\min\{M(Su, Tu, z, t), M(Au, Su, z, t), M(Bu, Tu, z, t), M(Su, Bu, z, t), M(Au, Tu, z, t)\})},$$

Using $(*)$, we have, $\boldsymbol{Bu = u}$.

Hence $\boldsymbol{Au = Bu = Su = Tu = u\ and\ u}$ is a common fixed point of $\boldsymbol{A, B, S\ and\ T}$.

Uniqueness follows easily.

Theorem 2.11: Let $\boldsymbol{A, B, S\ and\ T}$ be self maps of a fuzzy 2-metric space $(\boldsymbol{X, M}, *)$ satisfying (1),(2),(4) and the following conditions:

(7) pairs $(\boldsymbol{A},\boldsymbol{S})$ *and* $(\boldsymbol{B},\boldsymbol{T})$ satisfy a common E.A. property

If the range of S and T is a closed subset of X, then $\boldsymbol{A},\boldsymbol{B},\boldsymbol{S}$ *and* $\boldsymbol{T}$ have a unique common fixed point in X.

Proof: Suppose that $(\boldsymbol{A},\boldsymbol{S})$ *and* $(\boldsymbol{B},\boldsymbol{T})$ satisfy a common E.A. property. Then there exists a sequences $\{\boldsymbol{x_n}\}$ *and* $\{\boldsymbol{y_n}\}$ *in* $\boldsymbol{X}$ such that;

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u \text{ for some } u\epsilon X.$$

Since $\boldsymbol{S}(\boldsymbol{X})$ *and* $\boldsymbol{T}(\boldsymbol{X})$ are closed subsets of X, we obtain $\boldsymbol{u}=\boldsymbol{Sv}=\boldsymbol{Tw}$ *for some* $\boldsymbol{v},\boldsymbol{w}$ *in* $\boldsymbol{X}$. From (6),

$$M(Av, By_n, z, t) \geq$$

$$\psi(\min\{M(Sv, Ty_n, z, t), M(Av, Sv, z, t), M(By_n, Ty_n, z, t), M(Sv, By_n, z, t), M(Av, Ty_n, z, t)\}),$$

Letting $\boldsymbol{n}\to\infty$ and, by $(*)$, we have, $\boldsymbol{u}=\boldsymbol{Av}=\boldsymbol{Sv}=\boldsymbol{Tw}$.

The rest of the proof follows from the Theorem 2.4.

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