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# The commuting graphs on groups $D_{2n}$ and $Q_n$

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**Abstract**

**Given group $G$, the commuting graph of $G$, is defined as the graph with vertex set $G - Z(G)$, and two distinct vertices $x$ and $y$ are connected by an edge, whenever they commute, that is $xy = yx$. In this paper we get some parameters of graph theory, as *independent number* and *clique number for groups* $D_{2n}, Q_n$.**

***Keywords**:* independent number, clique number, generalized quaternion group

1 Introduction

Given a finite group $G$, and a subset $X$ of $G$, the commuting graph associated with $X$ is $\varphi(X, G)$, a graph with vetex set $X$, and two distinct vertices $x$, $y$ are adjacent, whenever $xy = yx$. Many authors have studied $\varphi(G, X)$ for different choices of $G$ and $X$. In [4] and [5], Segev and Seits apply the commuting graph, with $G$ a nonabelian simple group and $X = G - \{1\}$. The non-commuting graph of a group $G$, denoted by $\triangle(G)$, is the complement of $\varphi(X, G)$, with $X = G - Z(G)$. There are some papers on non-commuting graphs of a group, for instance see [1, 3]. In this work we consider the commuting graph $\varphi(X, G)$, with $G$, a non-abelian finite group and $X = G - Z(G)$, and we denote $\varphi(X, G)$ by $\Gamma(G)$. In this paper we obtain *independent number*, *clique number* and minimum size of a vertex cover of non-commuting graphs on dihedral and *generalized quaternion groups*. The graph -teoretic notation and terminology are standard; see [2] for example.

The rest of this paper is organized as follows: The section 2 contains some notations and preliminaries. In section 3 we get *independent number*, *clique number* and minimum size of a vertex cover.

## 2 Notations and preliminaries

We consider simple graphs which are undirected, with no loops or multiple edges. For a graph $\Gamma$, we denote the vertex set and the edge set of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. We denote the number of vertices $\Gamma$ by $n(\Gamma)$. The degree of a vertex $v$ in $\Gamma$, is denoted by $d_\Gamma(v)$, is the number of edges incident to $v$ and if the graph is understood, then we denote $d_\Gamma(v)$ by $d(v)$. A graph $\Gamma$ is regular if the vertices of graph $\Gamma$ are of the same degree. A subset $X$ of the vertices of $\Gamma$ is called a clique, if the induced subgraph of $X$ is a complete graph. The maximum size of a clique in a graph $\Gamma$ is called the clique number of $\Gamma$ and denoted by $w(G)$. A subset $X$ of the vertices of $\Gamma$ is called an independent set if the induced subgraph on $X$ has no edges. The independence number of $\Gamma$ is the maximum size of an independent set of vertices and is denoted by $\alpha(\Gamma)$. A vertex cover of a graph $\Gamma$ is a set $Q \subseteq V(\Gamma)$ that contains at least one endpoint of every edge. The minimum size of a vertex cover is denoted by $\beta(G)$. Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are said to be isomorphic(is denoted $\Gamma_1 \cong \Gamma_2$) whenever there exists one-to-one onto mapping $\phi : V_1 \to V_2$ such that, for all $u, v \in E_1$ we have: $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$. The wreath product of two graphs $\Gamma_1$ and $\Gamma_2$, written $\Gamma_1[\Gamma_2]$, is given as follows: The vertices of $\Gamma_1[\Gamma_2]$ are all pairs $(x, y)$ where $x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$ and edges of $\Gamma_1[\Gamma_2]$ are the pairs $\{\{(x_1, y_1), (x_1, y_2)\} : \{y_1, y_2\} \in E(\Gamma_2)\}$ together with $\{\{(x_1, y_1), (x_2, y_2)\} : \{x_1, x_2\} \in E(\Gamma_1)\}$.

## 3 The commuting graph of groups $D_{2n}$ and $Q_n$

Throughout this section, let $D_{2n} = \langle a, b : b^2 = a^n = 1, b^{-1}ab = a^{-1}\rangle$ and $Q_n = \langle c, d : d^4 = c^{2^{n-1}} = 1, d^2 = c^{2^{n-2}}, d^{-1}cd = c^{-1}\rangle$ denote the dihedral group and generalized quaternion group, respectively.

Let $[G : Z(G)] = m(m \geq 4)$, and $T = \{1, x_1, x_2, \dots, x_{m-1}\}$ be a transversal of $Z(G)$ in $G$. It is clear that every two element of the coset $x_iZ(G)$, $1 \leq i \leq m-1$, commute. thus, every two elements of these cosets are adjacent. We associate to the commuting graph $\Gamma(G)$ of a group $G$, the induced subgraph $\Gamma^u(G)$ as follows: The vertex set of $\Gamma^u(G)$ is $T - \{1\} = \{x_1, x_2, \dots, x_{m-1}\}$ , and two vertices $x_i$ and $x_j$, $1 \leq i, j \leq m-1$ are adjacent, if and only if $x_ix_j = x_jx_i$.

**Theorem 1.** *Let $G$ be a non-abelian group. Then $\Gamma(G) \cong \Gamma^u(G)[K_l]$, such that $l = |Z(G)|$.*

**Proof.** Let $Z(G) = \{a_1, a_2, \dots, a_l\}$ and $T = \{1, x_1, x_2, \dots, x_m\}$ be a transversal of $Z(G)$ in $G$. Then the set of cosets $x_iZ(G) = \{x_ia_1, x_ia_2, \dots, x_ia_l\}$ , $1 \leq i \leq m-1$, parties $V(\Gamma(G))$. Let $V(K_l) = \{1, 2, \dots, l\}$. Then map $\varphi : V(\Gamma(G)) \to V(\Gamma^u(G)[K_l])$, $x_ia_j \to (x_i, j)$, $1 \leq i \leq m-1$, $1 \leq j \leq l$ is a isomorphism, and this completes the proof.

**Lemma 1.** *[2] Let $\Gamma$ is a graph. Then $\alpha(\Gamma) + \beta(\Gamma) = n(\Gamma)$.*

**Lemma 2.** *Let $G$ be a non-abelian finite group. Then*

$$w(\Gamma(G)) = w(\Gamma^u(G))|Z(G)|.$$

**Proof.** Let $A$ is a clique on $\Gamma^u(G)$, such that $|A| = w(\Gamma^u(G))$. Then for each two cosets $xZ(G)$ and $yZ(G)$ such that $x, y \in A$, if $a \in xZ(G)$ and $b \in yZ(G)$, then $a$ and $b$ are adjacent.

Hence $X = \bigcup_{x\in A} xZ(G)$ is a clique and $|X| = |A||Z(G)| = w(\Gamma^u(G))|Z(G)|$. Thus $w(\Gamma(G)) \geq w(\Gamma^u(G))|Z(G)|$. Now let $A'$is a clique of $\Gamma(G)$, such that $|A'| = w(\Gamma(G))$. If $A' \cap x_i Z(G) \neq \emptyset$, $1 \leq i \leq m-1$, then $x_i Z(G) \subseteq A'$. Hence there exists $x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in T \setminus \{1\}$ such that $A' = \bigcup_{j=1}^{k} x_{i_j} Z(G)$. It is clear that $|A'| = k|Z(G)|$, Also, the set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is a clique on $\Gamma^u(G)$. Thus $k \leq w(\Gamma^u(G))$ and it follows that

$$w(\Gamma(G)) = |A'| \leq w(\Gamma^u(G))|Z(G)|.$$

Hence $w(\Gamma(G)) = w(\Gamma^u(G))|Z(G)|$.

**Lemma 3.** *Let $G$ be a non-abelian finite group. Then*

$$\alpha(\Gamma(G)) = \alpha(\Gamma^u(G)).$$

**Proof.** If $A$ is an independent set on $\Gamma(G)$, then $|A \cap x_i Z(G)| \leq 1$, $x_i \in T \setminus \{1\}$, because every two elements of a coset $x_i Z(G)$, $x_i \in T \setminus \{1\}$ are adjacent. For an independent set $A$ of $\Gamma(G)$, we associate an independent set $A'$ of $\Gamma^u(G)$ contain elements $x_i$ of $T \setminus \{1\}$ such that $A \cap x_i Z(G) \neq \emptyset$. Since $|A \cap x_i Z(G)| \leq 1$, $|A| = |A'|$. Thus $\alpha(\Gamma(G)) \leq \alpha(\Gamma^u(G))$. On the other hand, $\alpha(\Gamma^u(G)) \leq \alpha(\Gamma(G))$, because $\Gamma^u(G)$ is a subgraph of $\Gamma(G)$. Hence $\alpha(\Gamma(G)) = \alpha(\Gamma^u(G))$.

**Lemma 4.** *Let $G$ be a non-abelian finite group. Then*

$$\beta(\Gamma(G)) = \beta(\Gamma^u(G)) + (m-1)(|Z(G)|-1).$$

**Proof.** By lemma 2, we have:

$$\beta(\Gamma(G)) = (m-1)|Z(G)| - \alpha(\Gamma(G))$$

$$= (m-1)|Z(G)| - \alpha(\Gamma^u(G))$$

$$= (m-1)|Z(G)| + \beta(\Gamma^u(G)) - (m-1)$$

$$= \beta(\Gamma^u(G)) + (m-1)(|Z(G)|-1)$$

Now by [6. Lem 3.2] we get the following lemma.

**Lemma 5.** *For every odd $n$, two graphs $\Gamma(D_{2n})$ and $\Gamma^u(D_{2(2n)})$ are isomorphic.*

**Proposition 1.** *For every even natural number $n$, the following holds:*

*(i)* $w(\Gamma^u(D_{2n})) = \frac{n}{2} - 1$

*(ii)* $\alpha\left(\Gamma^u(D_{2n})\right) = \frac{n}{2} + 1$

*(iii)* $\beta\left(\Delta^u\ (D_{2n})\right) = \frac{n}{2} - 2$

**Proof.** *(i)*We have $Z(D_{2n}) = \left\{1, a^{\frac{n}{2}}\right\}$ and the set $T = \left\{1, a, a^2, \dots, a^{\frac{n}{2}-1}, b, ba, \dots, ba^{\frac{n}{2}}\right\}$ is a transversal of $Z(D_{2n})$ in $D_{2n}$. The set $A = \left\{a^i : 1 \le i \le \frac{n}{2} - 1\right\}$ is a clique of $\Gamma^u(D_{2n})$ and if $B$ be a clique of $\Gamma^u(D_{2n})$, then $B \subseteq A$. Thus $|A| = \frac{n}{2} - 1 = w\left(\Gamma^u(D_{2n})\right)$.

(ii) For any $j$ that $1 \le j \le \frac{n}{2} - 1$, the set $A_j = \left\{a^j, b, ba, \dots, ba^{\frac{n}{2}}\right\}$ is an independent set of $\Gamma^u(D_{2n})$ and each two elements of the set $\left\{a^i : 1 \le i \le \frac{n}{2} - 1\right\}$ are adjacent. Thus, for any $j$ that $1 \le j \le \frac{n}{2} - 1$, $\left|A_j\right| = \alpha\left(\Gamma^u(D_{2n})\right) = \frac{n}{2} + 1$ . Finally, the relation (iii) get from (ii) and Lemma 2.

**Theorem 2.** *For every natural number $n$, the followings hold:*

(1) If $n$ be even, then

*(i)* $w\left(\Gamma(D_{2n})\right) = n - 2$

*(ii)* $\alpha\left(\Gamma(D_{2n})\right) = \frac{n}{2} + 1$

*(iii)* $\beta\left(\Delta\ (D_{2n})\right) = \frac{3}{2}n - 3$

(2) If $n$ be odd, then

*(i)* $w\left(\Gamma(D_{2n})\right) = n - 1$

*(ii)* $\alpha\left(\Gamma(D_{2n})\right) = n + 1$

*(iii)* $\beta\left(\Gamma(D_{2n})\right) = n - 2$

**Proof.** (1) The proof follows by Lemmas 2, 3, 4 and Proposition 1.

(2) The proof follows by Lemmas 5 and Proposition 1.

**Proposition 2.** *For generalized quaternion group $Q_n$, the following hold:*

*(i)* $w\left(\Gamma^u(Q_n)\right) = 2^{n-2} - 1$

*(ii)* $\alpha\left(\Gamma^u(Q_n)\right) = 2^{n-2} + 1$

*(iii)* $\beta\left(\Gamma^u(Q_n)\right) = 2^{n-2} - 2$

**Proof.** *(i)* We have $Z(Q_n) = \left\{1, c^{2^{n-1}}\right\}$ and the set $T = \left\{1, c, c^2, \dots, c^{2^{n-2}-1}, d, dc, \dots, dc^{2^{n-2}-1}\right\}$ is a transvernal of $Z(Q_n)$ in $Q_n$. The set $A = \left\{c^i : 1 \le i \le 2^{n-2} - 1\right\}$ is a clique of $\Gamma^u(Q_n)$ and if $B$ be a clique of $\Gamma^u(Q_n)$, then $B \subseteq A$. Thus $|A| = 2^{n-2} - 1 = w\left(\Gamma^u(Q_n)\right)$.

*(ii)* For any $j$ that $1 \leq j \leq 2^{n-2} - 1$, the set $A_j = \{c^j, d, dc, \ldots, dc^{2^{n-2}-1}\}$ is an independent set of $\Gamma^u(Q_n)$ and each two elements of the set $\{c^i : 1 \leq i \leq 2^{n-2} - 1\}$ are adjacent. Thus, for any $j$, $1 \leq j \leq 2^{n-2} - 1$, $|A_j| = \alpha(\Gamma^u(Q_n)) = 2^{n-2} + 1$. The relation *(iii)* follows from Lemma 1 and *(ii)*.

Now by Lemmas 2, 3, 4 and Proposition 2 we conclude the following theorem.

**Theorem 3.** *For generalized quaternion group $Q_n$, the following hold:*

*(i)* $w(\Gamma(Q_n)) = 2^{n-1} - 2$

*(ii)* $\alpha(\Gamma(Q_n)) = 2^{n-2} + 1$

*(iii)* $\beta(\Gamma(Q_n)) = 3 \times 2^{n-2} - 3$

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