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Approximate Solutions of the Q-discrete Burgers Equation

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Abstract

Q-difference equations are a class of non-classical models. In this study, a combined method which has the merits of the varitional iteration method and the Adomian decomposition method is proposed. Then, the method is applied to a *q*-Burgers equation and approximate solutions are obtained.

Keywords: Variational iteration method; Adomian decomposition method; *Q*-Burgers equation

1. Introduction

The *q*-derivative is a deformation of the classical derivative and usefully employed to describe nonlinear phenomena in quantum diffusion systems, i.e., the non-equilibrium system [1], *q*-soliton [2, 3], fractal geometry [4], variational *q*-calculus [5]. Then the *q*-difference method is used to modeling the nonlinear problems and now to find approximate solutions of such models is undertaking. Some analytical methods have been proposed [6-9].

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The variational iteration method (VIM) [10-12] and the Adomian decomposition method (ADM) [13, 14] have their own merits and they have been two often used methods in the past ten years, i.e., for initial value problems of differential equations [15], Fuzzy equation [16] and fractional calculus [17].

Recently, the variational iteration method is successfully extended to *q*-difference equations [10-12]. In this study, the ADM [13, 14] is a famous linearization technique which is used to handle nonlinear terms of the governing equations and make the VIM more efficient. A q-Burgers equation is illustrated as an example.

2. Preliminaries

In this section, some properties of the *q*-calculus are introduced.

For $0<q<1,$ let T_q be the time scale: $T_q=\{q^n:n\in Z\}\cup\{0\}$ where Z is the set of integers.

2.1 The q - integration of $f(t)$ is defined by

$$\int_0^x f(t)d_q t=(1-q)x\sum_{n=0}^{\infty}q^n f(q^n x). \tag{1}$$

Further, let $f(x;y;...)$ be a multivariable real continuous function. Then the q -derivative and the partial q -derivative are defined as

$$\begin{aligned}
&\frac{d}{d_q x}f(x)=\frac{f(qx)-f(x)}{(q-1)x},\\
&\frac{\partial f(x;y;...)}{\partial_q x}=\frac{f(qx;y;...)-f(x;y;...)}{(q-1)x},\\
&\frac{\partial^n f(x;y;...)}{\partial_q x^n}=\underbrace{\frac{\partial}{\partial_q x}\cdots\frac{\partial}{\partial_q x}}_{n}f(x;y;...).
\end{aligned} \tag{2}$$

For $x=0,$ $\partial f(x;y;...)/\partial_q x\big|_{x=0}=\lim\limits_{n\to\infty}\frac{f(q^n;y;...)-f(0;y;...)}{q^n}.$

2.2 The Leibniz rule for a q -derivative of a product of two functions is

$$\frac{d}{d_q x}[f(x)g(x)]=g(qx)\frac{d}{d_q x}[f(x)]+f(x)\frac{d}{d_q x}[g(x)]. \tag{3}$$

2.3 q -Integration by parts holds

$$\int_a^b g(qx)\frac{d}{d_q x}f(x)d_q x = f(x)g(x)|_a^b - \int_a^b f(x)\frac{d}{d_q x}g(x)d_q x. \tag{4}$$

More results and recent developments in this area are available in [18-20].

3. The VIM for *q*-difference equations

Following the VIM's rule: (a) establishing the correction functional; (b) identifying the Lagrange multipliers; (c) determining the initial iteration. For a nonlinear *q*-difference equation

$$\frac{d^m u}{d_q t^m} + f(t,u)=0, \tag{5}$$

one can first construct the correction functional by using (1),

$$u_{n+1} = u_n + \int_0^t \lambda(t, q^m\tau)[\frac{d^m u_n}{d_q \tau^m} + f(\tau, u_n)]d_q\tau. \tag{6}$$

Then, a *q*-Lagrange multiplier can be optimally determined by the variational *q*-calculus [5]. Three cases have been discussed in [9, 21, 22]. After determination of the initial iteration u_0 from the *q*-Taylor series [19], the approximate solution tends to the exact solution of (5) for $n \to \infty$.

Now if $f(t,u)$ has a nonlinear term, we consider to using the famous linearized technique, the Adomian series to expand the $f(t,u)$ approximately. We don't given the detail expressions here. Readers who feel interested in the ADM are referred to the original idea [13, 14] and the recent development [23-30]. For the *q*-difference equations (5), analysis of the existence and the uniqueness of the solution can be found in [20].

4. Approximate solutions of the *q*-discrete Burgers equations

The Burgers equation is the simplest nonlinear generalization of the diffusion equation. It occurs in various areas of applied mathematics, such as modeling of gas dynamics and traffic flow. The equation of motion in one dimension has the following form

$$\frac{\partial u}{\partial t} + uu_x = v\frac{\partial^2 u}{\partial x^2}$$

where $u=u(x,t)$ is the velocity and v is the viscosity coefficient.

In order to describe a non-equilibrium distribution, for $0<q<1$, let T_q be the time scale:

$T_q=\{q^n: n\in Z\}\cup\{0\}$ where Z is the set of positive integers. We consider a *q*-discrete Burgers equation as follows

$$\frac{\partial u}{\partial_q t}+uu_x=\frac{\partial^2 u}{\partial x^2},\ u(x,0)=\sin(2\pi x),\ \ u(0,t)=u(1,t)=0, 0\le x\le 1. \tag{7}$$

We revisit the identification of the Lagrange multiplier [9]. First, construct the correction function as

$$u_{n+1}=u_n+\int_0^t \lambda(t,q\tau)[\frac{\partial u_n}{\partial_q \tau}+u_n u_{n,x}-u_{n,xx}]d_q\tau.$$

Then, through the integration by parts and the term being handled as restricted variation, one can derive

$$\delta u_{n+1}=\delta u_n+\delta\int_0^t \lambda(t,q\tau)[\frac{\partial u_n(x,\tau)}{\partial_q\tau}]d_q\tau=(1+\lambda(t,\tau)|_{\tau=t})\delta u_n-\int_0^t\frac{\partial\lambda(t,\tau)}{\partial_q\tau}\delta u_n(x,\tau)d_q\tau$$

so that

$$\begin{cases}1+\lambda(t,\tau)|_0^t=0,\\ \dfrac{\partial\lambda(t,\tau)}{\partial_q\tau}=0.\end{cases} \tag{8}$$

Namely, the simplest Lagrange multiplier can be identified as $\lambda(t,\tau)=-1$ and $\lambda(t,q\tau)=-1$.

As a result, substituting this result into (8), we obtain the variational iteration formula

$$u_{n+1}=u_n-\int_0^t[\frac{\partial u_n}{\partial_q\tau}+u_n u_{n,x}-u_{n,xx}]d_q\tau. \tag{9}$$

On the other hand, let $u_n=\sum_{i=0}^{n} v_i$ and the nonlinear term uu_x can be approximately expanded as the Adomian polynomials $\sum A_n$ [14]

$$A_0 = v_0 v_{0,x},$$
$$A_1 = v_0 v_{1,x} + v_1 v_{0,x},$$
$$A_2 = v_0 v_{2,x} + v_1 v_{1,x} + v_2 v_{1,x},$$
$$\vdots$$

Here the initial iteration value v_0 can be determined as $v_0 = u(x,0) = \sin(2\pi x)$. The above idea using the Adomian polynomials was proposed by Abbasbandy in [31].

The iteration formula now can be modified as

$$v_{n+1} = \int_0^t (v_{n,xx} - A_n) d_q \tau, v_0 = \sin(2\pi x).$$

Since Eq. (2) is a *q*-difference equation of first order, it is interesting to point out that the approximate solution here is the same as the one derived by the ADM. As a result, one can obtain v_i without any difficulty

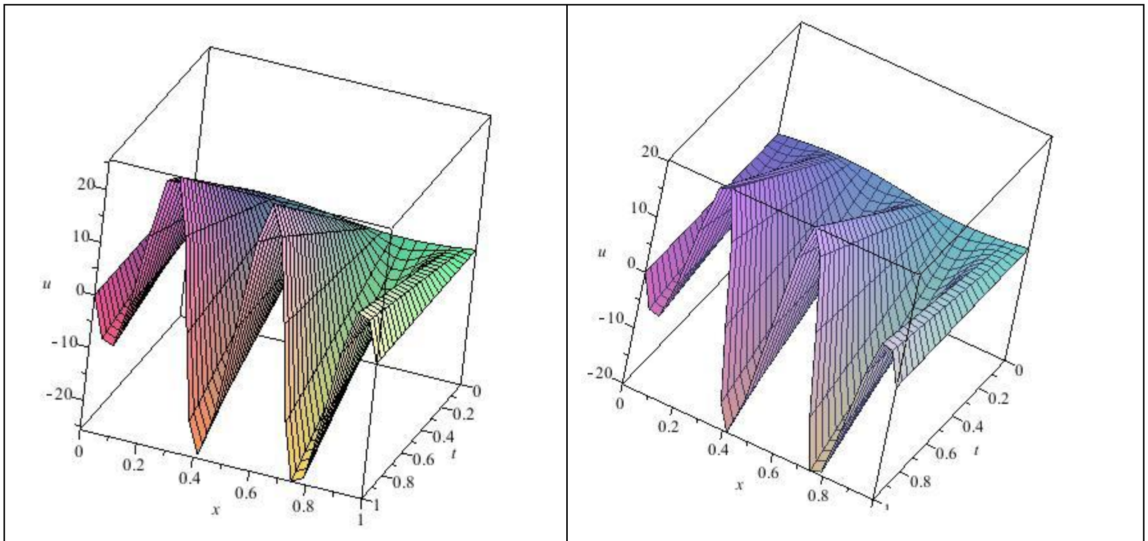
$$v_0 = \sin(2\pi x),$$
$$v_1 = -\sin(2\pi x)(2\pi\cos(2\pi x) - 1)\frac{t}{[1]_q!},$$
$$v_2 = -\sin(2\pi x)(1 - 4\pi^2 - 6\pi\cos(2\pi x) + 12\pi^2\cos(2\pi x)^2)\frac{t^2}{[2]_q!},$$
$$\vdots$$

where $[1]_q! = 1$ and $[2]_q! = [1]_q[2]_q = 1 + q$.

If the second order approximation is enough, we can derive the approximate solution as

$$u \cong v_0 + v_1 + v_2, \tag{10}$$

The following figures illustrate the approximate solutions at various q $(0 < q \le 1)$.



$q=0.5$	$q=1$

Fig. 1 The *q*-discrete Burgers' flow for $q=0.5$ and $q=1$

5. Conclusions

In this study, a combined method of the VIM and the ADM is proposed for nonlinear *q*-difference equations. Then it is used to approximately solve the *q*-Burgers equation on time scale. It follows two main steps: identification of the *q*-Lagrange multipliers and linearization of the nonlinear terms by the ADM.

The obtained solutions have a *q* parameter and they are illustrated for different values of *q*. The presented method is direct and has the merits of both the VIM and the ADM which also can be used in other initial value problems of the *q*-difference equations.

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