

ON THE SOLVING NONLINEAR APPROXIMATE LONG WAVE EQUATIONS

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Abstract

In this letter, He's Variational Iteration Method (VIM) is implemented for solving the nonlinear Whitham-Broer-Kaup partial differential equations in the special case is named approximate long wave equations (ALW).

This method is based on Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method creates a sequence which those obtained by the Adomian decomposition method (ADM). The work confirms that the VIM method is superior and very faster to the ADM .

Keywords: Whitham-Broer-Kaup equations, Approximate Long Wave equations, Variational Iteration Method, Adomian Decomposition Method.

1. Introduction

Most scientific problems such as Whitham-Broer-Kaup (WBK) equations are inherently of nonlinearity. We know that except a limited number of these problems, Most of them do not have

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analytical solution. Therefore, these nonlinear equations should be solved using other methods. In this article, VIM is used to solve nonlinear equations of Whitham-Broer-Kaup with the initial conditions $u(x,0) = F(x)$, $v(x,0) = G(x)$ for the first time:

$$\begin{cases} u_t + uu_x + v_x + \beta u_{xx} = 0, \\ v_t + (uv)_x + \alpha u_{xxx} - \beta v_{xx} = 0. \end{cases} \quad (1.1)$$

If $\alpha = 0$, $\beta \neq 0$, then the system of Eqs. (1.1) becomes classical long wave equation that describe shallow water wave with dispersive [13]. If $\alpha=0$, $\beta=1/2$, then the system of Eqs. (1.1) can be reduced to the approximate long wave (ALW) equations [13]. In this paper we will focus on finding analytical approximate and exact traveling wave solution above equations using the variational iteration method. Also we compare the result with given solutions using ADM [3, 11].

2. Variational Iteration Method

In this section, we use the following non-homogeneous system of partial differential equations

$$\begin{cases} L_1 u(x,t) + N_1(u(x,t), v(x,t)) = f(x,t), \\ L_2 u(x,t) + N_2(u(x,t), v(x,t)) = g(x,t), \end{cases} \quad (1.2)$$

to illustrate the basic idea of the variational iteration method [1,2,4,5,6,9,10,12]. In the above system of equations L_1 and L_2 are linear differential operators with respect to t and N_1 and N_2 are nonlinear operators and $f(x,t)$, $g(x,t)$ are some given functions. According to the variational iteration method, we can construct a correct functional as follows [1,2,4,5,6,9,10,12]:

$$\begin{cases} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\tau) \{L_1(u_n(x,\tau) + \tilde{v}_n(x,\tau)) + N_1(u_n(x,\tau), \tilde{v}_n(x,\tau)) - f(x,\tau)\} d\tau, \\ v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\tau) \{L_2(v_n(x,\tau) + \tilde{u}_n(x,\tau)) + N_2(\tilde{u}_n(x,\tau), v_n(x,\tau)) - g(x,\tau)\} d\tau, \end{cases} \quad (2.2)$$

where λ_1 and λ_2 are general Lagrange multipliers, which can be identified optimally via variational theory [1,2,4,5,6,7,8,9,10,12]. The second terms on the right-hand side in (2.2) are called the correction and the subscript n denotes the n th-order approximation. Under a suitable restricted variational assumptions (i.e. \tilde{u}_n and \tilde{v}_n are considered as a restricted variation), we may assume that the above correctional functionals are stationary (i.e. $\delta \tilde{u}_n = 0$ and $\delta \tilde{v}_n = 0$), then the Lagrange multipliers are identified. Now we may start the procedure with the given initial approximation and using the above iteration formulas to obtain the approximate solutions.

3. Implementation of the method

We first consider the application of VIM to WBK Eqs.(1.1) with the initial conditions of

$$u(x,0) = w - 2Bk \coth(k\varepsilon), \quad v(x,0) = -2B(B + \beta)k^2 \operatorname{csch}^2(k\varepsilon), \quad (1.3)$$

where $B = \sqrt{\alpha + \beta^2}$ and $\varepsilon = x + x_0$ and x_0, k, w are arbitrary constants.

To solve Eqs. (1.1) with the initial conditions (1.3) by means of VIM , we consider the correction functional in t-direction in the following from:

$$u_{n+1} = u_n + \int_0^t \lambda_1(\tau) \{u_{n_\tau} + \tilde{u}_n \tilde{u}_{n_x} + \tilde{v}_{n_x} + \beta \tilde{u}_{n_{xx}}\} d\tau, \quad (2.3)$$

$$v_{n+1} = v_n + \int_0^t \lambda_2(\tau) \{v_{n_\tau} + (\tilde{u}_n \tilde{v}_n)_x + \alpha \tilde{u}_{n_{xxx}} - \beta \tilde{v}_{n_{xx}}\} d\tau,$$

where λ_1 and λ_2 are general lagrange multipliers which its optimal value is found by using variational theory. The value of \tilde{u}_0 and \tilde{v}_0 are initial approximations and must be chosen suitably and \tilde{u}_n and \tilde{v}_n are the restricted values i.e. $\delta \tilde{u}_n = 0$ and $\delta \tilde{v}_n = 0$.

Now we have

$$\delta u_{n+1} = \delta u_n + \delta \int_0^t \lambda_1(\tau) \{u_{n_\tau} + \tilde{u}_n \tilde{u}_{n_x} + \tilde{u}_{n_x} + \beta \tilde{u}_{n_{xx}}\} d\tau, \quad (3.3)$$

$$\delta v_{n+1} = \delta v_{n+1} + \delta \int_0^t \lambda_2(\tau) \{v_{n_\tau} + (\tilde{u}_n \tilde{v}_n)_x + \alpha \tilde{u}_{n_{xxx}} - \beta \tilde{v}_{n_{xx}}\} d\tau,$$

to find the optimal value of λ_i , we have $(\delta u_n(0)=0)$

$$\delta u_{n+1} = \delta u_n + \int_0^t \lambda_1(\tau) \{\delta u_{n_\tau}\} d\tau = (1 + \lambda_1(\tau)|_{\tau=t} \delta u_n - \int_0^t \lambda_1(\tau) \delta u_n d\tau = 0, \quad (4.3)$$

$$\delta v_{n+1} = \delta v_n + \int_0^t \lambda_2(\tau) \{\delta v_{n_\tau}\} d\tau = (1 + \lambda_2(\tau)|_{\tau=t} \delta v_n - \int_0^t \lambda_2(\tau) \delta v_n d\tau = 0.$$

Therefore , the stationary conditions are obtained in the following form:

$$\begin{aligned} \lambda_1'(\tau) &= 0, & 1 + \lambda_1(\tau) &= 0|_{\tau=t}, \\ \lambda_2'(\tau) &= 0, & 1 + \lambda_2(\tau) &= 0|_{\tau=t}. \end{aligned} \quad (5.3)$$

Therefore the Lagrange multipliers is defined as the following form:

$$\lambda_1(\tau) = -1, \quad \lambda_2(\tau) = -1. \quad (6.3)$$

Substituting Eq. (6.3) into the correctional functional Eq. (4.3) we get the following iteration formulas :

$$u_{n+1} = u_n - \int_0^t \{u_{n_\tau} + u_n u_{n_x} + v_{n_x} + \beta u_{n_{xx}}\} d\tau, \quad (7.3)$$

$$v_{n+1} = v_n - \int_0^t \{v_{n_\tau} + (u_n v_n)_x + \alpha u_{n_{xxx}} - \beta v_{n_{xx}}\} d\tau.$$

We start with initial approximations $u_0(x, t)=u(x,0)$ and $v_0(x, t)=v(x, 0)$ and by the above formulas , we can obtain the $u_1(x, t)$ and $v_1(x, t)$ as follows :

$$u_0(x, t) = \omega - 2Bk \coth(k(x+x_0)),$$

$$\begin{aligned}
 v_0(x, t) &= -2Bk^2(B+\beta) \operatorname{csch}^2(k(x+x_0)), \\
 u_1(x, t) &= \omega - 2Bk \operatorname{coth}(k(x+x_0)) \\
 &\quad + 2Bk^2t[\omega - 2Bk \operatorname{coth}(k(x+x_0))] [1 - \operatorname{coth}^2(k(x+x_0))] \\
 &\quad - 4Bk^3t(B+\beta) \operatorname{csch}^2(k(x+x_0) \operatorname{coth}(k(x+x_0))) \\
 &\quad - 4\beta Bk^3t \operatorname{coth}(k(x+x_0))[1 - \operatorname{coth}^2(k(x+x_0))], \\
 v_1(x, t) &= -2k^2(B+\beta) \operatorname{coth}^2(k(x+x_0)) \\
 &\quad - 4B^2k^4t(B+\beta)A \operatorname{coth}^2(k(x+x_0)) \\
 &\quad - 4Bk^3t(B+\beta) \operatorname{csch}^2(k(x+x_0) \operatorname{coth}(k(x+x_0))) \\
 &\quad \times [\omega - 2Bk \operatorname{coth}(k(x+x_0))] \\
 &\quad - \alpha t [4Bk^4 A^2 - 8Bk^4 A \operatorname{coth}^2(k(x+x_0))] \\
 &\quad - 8\beta Bk^4t(B+\beta) \operatorname{csch}^2(k(x+x_0) \operatorname{coth}^2(k(x+x_0))) \\
 &\quad + 4\beta Bk^4t(B+\beta)A \operatorname{csch}^2(k(x+x_0)), \\
 &\vdots
 \end{aligned}$$

Where, $A=1-\operatorname{coth}^2(k(x+x_0))$.

In the same manner the rest of components of the iteration formulas (7.3) can be obtained using MATLAB software.

Exact solutions of WBK Eqs .(1.1) are as follows :

$$u(x, t) = \omega - 2Bk \operatorname{coth}(k(\xi - \omega t)), \tag{8.3}$$

$$v(x, t) = -2B(B+\beta)k^2 \operatorname{csch}^2(k(\xi - \omega t)). \tag{9.3}$$

where $B = \sqrt{\alpha + \beta^2}$ and $\xi = x + x_0$, these solutions are constructed by Xie and et. al. [13].

For comparing results between VIM and exact solution, error ($u(x, t) = |u_{Exact} - u_{VIM}|$) and error ($v(x, t) = |v_{Exact} - v_{VIM}|$) are calculated and shown in Table 1 (a and b).

Table 1.a

Compare results of $u(x, t)$ for WBK equations by VIM ($u_{VIM} = u_1$) with exact result .

$ u_{Exact} - u_{VIM} $					
$x_i \setminus t_i$	0.2	0.4	0.6	0.8	1
0.2	3.4575e-09	1.3832e-08	3.1124e-08	5.5338e-08	8.6474e-08
0.4	3.2495e-09	1.2999e-08	2.9251e-08	5.2008e-08	8.1271e-08
0.6	3.0569e-09	1.2229e-08	2.7517e-08	4.8924e-08	7.6452e-08
0.8	5.8782e-09	1.1514e-08	2.5909e-08	4.6065e-08	7.1984e-08
1	2.7123e-09	1.0850e-08	2.4416e-08	4.3410e-08	6.7835e-08

Table 1.b

Compare results of $v(x,t)$ for WBK equations by VIM($v_{VIM}=v_I$) with exact result .

$ V_{Exact}-V_{VIM} $					
$x_i \backslash t_i$	0.2	0.4	0.6	0.8	1
0.2	3.7154e-09	1.4864e-08	3.3447e-08	5.9470e-08	9.2934e-08
0.4	3.4380e-09	1.3754e-08	2.0950e-08	5.5030e-08	3.5995e-08
0.6	3.1859e-09	1.2754e-08	2.8681e-08	5.0994e-08	7.9689e-08
0.8	2.9564e-09	1.1827e-08	2.6614e-08	4.7320e-08	7.3947e-08
1	2.7471e-09	1.0990e-08	2.4730e-08	4.3969e-08	6.8711e-08

1.3 Approximate long wave equations (ALW).

If $\alpha=0$ and $\beta=\frac{1}{2}$, then WBK Eqs .(1.1) can be reduced to the approximate long wave (ALW)

equations in shallow water

$$\begin{cases} u_t + uv_x + v_x + \frac{1}{2}u_{xx} = 0, \\ u_t + (uv)_x - \frac{1}{2}v_{xx} = 0, \end{cases} \tag{10.3}$$

with the initial conditions :

$$u(x, 0) = \omega - k \coth(k\xi), \quad v(x, 0) = -k^2 \operatorname{csch}^2(k\xi). \tag{11.3}$$

If we substitute $\alpha=0$ and $\beta=0.5$ in the obtained solutions for WBK Eqs.(1.1), or directly solve the system of Eqs . (10.3) with above initial conditions (Eqs. (11.3)) by VIM , we can obtain $u_1(x, t)$ and $v_1(x, t)$ as follows:

$$\begin{aligned} u_0(x, t) &= \omega - k \coth(k(x+x_0)), \\ v_0(x, t) &= -k^2 \operatorname{csch}^2(k(x+x_0)), \\ u_1(x, t) &= \omega - k \coth(k(x+x_0)) \\ &\quad + k^2 t [\omega - k \coth(k(x+x_0))] [1 - \coth^2(k(x+x_0))] \\ &\quad - 2k^3 t \operatorname{csch}^2(k(x+x_0)) \coth(k(x+x_0)) \\ &\quad - k^3 t \coth(k(x+x_0)) [1 - \coth^2(k(x+x_0))] , \\ v_1(x, t) &= -k^2 \operatorname{csch}^2(k(x+x_0)) \\ &\quad - 2k^3 t \operatorname{csch}^2(k(x+x_0)) \coth(k(x+x_0)) \end{aligned}$$

$$\begin{aligned} & \times [\omega - k \coth(k(x+x_0))] \\ & - 2k^4 t \operatorname{csch}^2(k(x+x_0)) \coth^2(k(x+x_0)), \end{aligned}$$

in the same manner the rest of components of the iteration formulas (7.3) can be obtained using MATLAB software.

Exact silutions of ALW Eqs. (17) are as follows:

$$u(x, t) = \omega - k \coth(k(\xi - \omega t)), \tag{12.3}$$

$$v(x, t) = -k^2 \operatorname{csch}^2(k(\xi - \omega t)). \tag{13.3}$$

For comparing results between VIM and exact solution , error $(u(x, t) = |u_{Exact} - u_{VIM}|)$ and error $(v(x, t) = |v_{Exact} - v_{VIM}|)$ are calculated and shown in Table 3 (a and b).

Table 3.a

Compare results of $u(x,t)$ for ALW equations by VIM($u_{VIM}=u_I$) with exact result.

$ u_{Exact} - u_{VIM} $					
$x_i \backslash t_i$	0.2	0.4	0.6	0.8	1
0.2	8.9237e-10	3.5713e-09	8.0362e-09	1.4288e-08	2.2328e-08
0.4	8.3901e-10	3.3564e-09	7.5527e-09	1.3428e-08	2.0984e-08
0.6	7.8928e-10	3.1574e-09	7.1049e-09	1.2632e-08	1.9740e-08
0.8	7.4315e-09	2.9729e-09	6.6897e-09	1.1894e-08	1.8586e-08
1	7.0032e-09	2.8016e-09	6.3041e-09	1.1208e-08	1.7515e-08

Table 3.b

Compare results of $v(x,t)$ for ALW equations by VIM($v_{VIM}=v_I$) with exact result.

$ v_{Exact} - v_{VIM} $					
$x_i \backslash t_i$	0.2	0.4	0.6	0.8	1
0.2	2.7916e-10	1.1168e-09	2.5131e-09	4.4682e-09	6.9825e-09
0.4	2.5831e-10	1.0334e-09	2.3254e-09	4.1346e-09	6.4612e-09
0.6	2.3973e-10	9.5762e-10	2.1549e-09	3.8314e-09	5.9874e-09
0.8	2.2213e-10	8.8863e-10	1.9997e-09	3.5554e-09	5.5560e-09
1	2.0640e-10	8.2571e-10	1.8581e-09	3.3036e-09	5.1625e-09

4. Conclusions

In this paper , He's variational iteration method has been successfully applied to find the solutions of nonlinear WBK in particular ALW equations . All the examples show that the results of the present method are in excellent agreement with those of exact solutions.

In our work , we use the MATLAB software to calculate the functions obtained from the variational iteration method.The results show that this method provides excellent approximations to the solution of this nonlinear system with high accuracy .

This new method accelerated the convergence to the solutions. Finally , it has been attempted to show the capabilities and wide – range applications of the variational iteration method .

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