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Numerical method for solving a kind of Volterra integral equation using differential transform method

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Abstract

In this paper we use differential transform method for solving nonlinear and linear Volterra integral equation with the kernel as $k(x-t)$ by using an efficient technique. We approximate the kernel of integral equation with Taylor series and make integral equation simpler by using some techniques that when we use differential transform method, we do not need difficult computation. Note that without this technique, solving integral equation by DTM method will be hard. Through some examples, we have shown the application of these techniques and differential transform method.

Keywords: Differential transform method; Volterra integral equation; Numerical method

1. Introduction

The concept of the differential transform was first proposed by Zhou [14], and its main application therein is solved both linear and nonlinear initial value problems in electric circuit analysis. The DTM gives exact values of the nth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytical Taylor

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series solutions of differential equations. Different applications of DTM can be found in [1-10, 13].

Let $u(x)$ be the function to be solved for, $f\ (x)$ is a given known function, and $k(x,t)$ a known integral kernel.The Volterra integral equation of the first kind is an integral equation of the form[13]:

$$f\ (x) = \int_a^x k(x,t)u(t)dt, \qquad (1.1)$$

and the Volterra integral equation of the second kind is an integral equation of the form:

$$u\ (x)\ =\ f(x)+\int_a^x k(x,t)u(t)dt, \qquad (2.1)$$

In this study we survey the kind of Volterra integral equation in the following form:

$$u(x) = f(x) + \int_0^x k(x-t)u(t)dt, \qquad (3.1)$$

We think that[13], this paper can be used to convey to students the idea that the DTM is a powerful tool for approximately solving linear and nonlinear Volterra integral equation of the first kind or the second kind that kernel is as $k(x-t)$. DTM evaluates the approximating solution and kernel by the finite Taylor series.

Several examples of linear and nonlinear Volterra integral equations are tested using DTM, and the obtained result that the DTM is very effective and simple.

2. Basic idea of differential transform method

The basic definitions and fundamental operations of differential transform are given in [1-3, 6-10, 13, and 14]. For convenience of the reader, we will present a review of the differential transform method. The differential transform of the derivative of a function is defined as follows:

$$F(k) =\frac{1}{k!}[\ \frac{d^k f(x)}{dx^k}]_{x=x_0} \qquad (1.2)$$

Where$f\ (x)$the original is function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-t)^k. \qquad (2.2)$$

From Eqs. (1.2) and (2.2), we get

$$f\ (x) =\sum_{k=0}^{\infty} \frac{(x-t)^k}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=x_0}. \qquad (3.2)$$

This implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are

Described by the transformed equations of the original function. In real applications, the function $f\ (x)$ is expressedby a finite series and Eq. (2.2) can be written as

$$f(x)=\sum_{k=0}^{n}F(k)(x-t)^{k}. \qquad (4.2)$$

Here n is decided by the convergence of natural frequency.

We achieve Taylor series of kernel.

Let$(x-t)=v\ and\ (x_0-t_0)=v_0$, we have obtain:

$$k(v)=k(v_0)+k(v_0)(v-v_0)+k(\acute{v_0})\frac{(v-v_0)^2}{2!}+\cdots \qquad (5.2)$$

We insert(5.2) in the equation (3.1):

$$u(x)=f(x)+\int_0^x(k(v_0)+k(v_0)(v-v_0)+\cdots)u(t)dt. \qquad (6.2)$$

Then,

We segregate$u(x)$into$u_0(x),u_1(x),\dots,u_n(x)$.

$$u_0(x)=f(x), \qquad (7.2)$$

$$u_1(x)=\int_0^x k(v_0)u(t)dt, \qquad (8.2)$$

$$u_2(x)=\int_0^x k(v_0)(v-v_0)u(t)dt,$$

$$u_3(x)=\int_0^x k(\acute{v_0})\frac{(v-v_0)^2}{2!}u(t)dt,$$

$$u_4(x)=\int_0^x k(\acute{v_0})\frac{(v-v_0)^3}{3!}u(t)dt,$$

$\ldots(9.2)$

By using differential equations (9.2), we obtain:

$$\frac{du_2(x)}{dx} = \frac{d}{dx}\int_0^x k(v_0)(v-v_0)u(t)dt,$$

$$\frac{d^2u_3(x)}{dx^2} = \frac{d^2}{dx^2}\int_0^x k(\acute{v_0})\frac{(v-v_0)^2}{2!}u(t)dt,$$

$$\frac{d^3u_4(x)}{dx^3} = \frac{d^3}{dx^3}\int_0^x k(\acute{v_0})\frac{(v-v_0)^3}{3!}u(t)dt,$$

$$\ldots \qquad (10.2)$$

Finally by using DTM equations(7.2), (8.2) and (10.2),

We will obtain $U_0(k), U_1(k), \ldots, U_n(k)$, and insert them in

$$U(k) = \sum_{i=0}^{n} U_i(k). \qquad (11.2)$$

In the following theorem, we find the differential transformation for two types of product of single-valued functions. These results are very useful on our approach for solving integral equations.

Theorem1. [13] Suppose that $U(k)$ and $G(k)$ are the differential transformations of the functions $u(x)$ and $g(x)$, respectively, then we have the following properties:

(a) $if f(x) = x, then$

$$F(k) = \delta(k-1) \to \begin{cases} 1 & if k = 1 \\ 0 & if k \neq 1 \end{cases} \quad (12.2)$$

(b)$if f(x) = \int_{x_0}^{x} g(t)dt, then$

$$F(k) = \frac{G(k-1)}{k}, F(0) = 0 \qquad (13.2)$$

(c) $if f(x) = \int_{x_0}^{x} g(t)u(t)dt, then$

$$F(k) = \sum_{l=0}^{k-1} G(l)\frac{U(k-l-1)}{k}, F(0) = 0 \qquad (14.2)$$

(d) $if f(x) = \frac{d^n u(x)}{dx^n}, then$

$$F(k) = (k+1)(k+2)\dots(k+n)U(k+n), \qquad (15.2)$$

(e) $If f(x) = \int_{x_0}^{x} u_1(t)u_2(t)\dots u_n(t)dt$, $then$

$$F(k) = \frac{1}{k}\sum_{k_{n-1}=0}^{k_n-1}\sum_{k_{n-2}=0}^{k_{n-1}}\dots\sum_{k_1=0}^{k_2} U_1(k_1)U_2(k_2-k_1)\dots U_n(k_n-k_{n-1}-1)$$

$$,F(0) = 0 \qquad (16.2)$$

Proof. The proof follows immediately from the definitions (*1.2*) and (*2.2*) and the operations of differential transformation given in [13].

Theorem 2. [*12*] $If f(x) = \int_0^x \int_0^{x_1}\dots\int_0^{x_n} f(t)dt$

$$Then, f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \qquad (17.2)$$

3. Applications and numerical results

In order to illustrate the advantages and the accuracy of the DTM for solving the nonlinear Volterra integral equation.

Example3.1. Consider the linear Volterra integral equation:

$$u(x) = x + \int_0^x (x-t)u(t)dt \; u(0) = 0 \;. \qquad (1.3)$$

Using differential of $u(x)$, we obtain:

$$\frac{du(x)}{dx} = 1 + \int_0^x u(t)dt \qquad (2.3)$$

Using properties of DTM, we have the following recurrence relation:

$$(k+1)\,U(k+1) = \delta(k) + \frac{U(k-1)}{k} \quad ,k \geq 0 \qquad (3.3)$$

Consequently, we find:

$U(0) = 0,$ $\qquad U(1) = 1,$ $\qquad U(2) = 0,$

$U(3) = \frac{1}{3!}$, $U(4) = 0,$ $U(5) = \frac{1}{5!}$,

$U(6) = 0,$ $U(7) = \frac{1}{7!}$, $U(8) = 0,$

…

Therefore, from (3.3), the solution of the integral equation (1.3) is given by

$u\ (x)= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots$

$u(x) = sinh(x)$

Example 3.2.[12]Consider the nonlinear Volterra integral equation:

$$u(x) = 1+\int_0^x (x-t)^2 u^2(t)dt. \tag{4.3}$$

Using differential of $u(x)$,we obtain:

$$\frac{du(x)}{dx} = 2\int_0^x (x-t)u^2(t)dt \quad ,$$

$$\frac{d^2u(x)}{dx^2} = 2\int_0^x u^2(t)dt,$$

Using properties of DTM, we have the following recurrence relation:

$$(k+1)(k+2)U(k+2) = 2\sum_{l=0}^{k-1} \frac{U(l)U(k-l-1)}{k}, \tag{5.3}$$

$U(0) = 1,$

$U(1) = 0,$

$U(2) = 0,$

$U(3) = \frac{1}{3},$

$U(4) = 0,$

$U(5) = 0,$

$U(6) = \frac{1}{90},$

…

Therefore, from (2.2), the solution of the integral equation (4.3) is given by

$u\ (x)$=1+$\frac{1}{3}x^3+\frac{1}{90}x^6+\cdots$

Example3.3. Let us consider the following nonlinear Volterra integral equation:

$$u(x)=x+\int_0^x \sinh(x-t)\,u^3(t)dt\ . \qquad (6.3)$$

We achieve Taylor series of kernel and we put it in the equation(6.3).

$$sinh(x-t)=(x-t)+\frac{(x-t)^3}{3!}+\frac{(x-t)^5}{5!}+\cdots$$

$$\sinh(x-t)\approx(x-t)+\frac{(x-t)^3}{3!}$$

$$u(x)=\ x+\int_0^x(x-t)u^3(t)dt+\int_0^x\frac{(x-t)^3}{3!}u^3(t)dt,$$

We segregate $u(x)$in to$u_0(x),u_1(x),u_2(x),$

$$u_0(x)=x,$$

$$u_1(x)=\int_0^x(x-t)u^3(t)dt,$$

$$u_2(x)=\int_0^x\frac{(x-t)^3}{3!}u^3(t)dt,$$

By using differential we approximating kernels and in a final manner we utilize differential transform.

$$\frac{du_1(x)}{dx}=\int_0^x u^3(t)dt,$$

$$\frac{d^3u_2(x)}{dx^3}=\int_0^x u^3(t)dt,$$

$$U_0(k)=\delta(k-1), \qquad (3.7)$$

$$(k+1)U_1(k+1)=\sum_{k_1=0}^{k-1}\sum_{l=0}^{k_1}\frac{U(l)U(k_1-l)U(k-k_1-1)}{k},(3.8)$$

$$(k+1)(k+2)(k+3)U_2(k+3)=\sum_{k_1=0}^{k-1}\sum_{l=0}^{k_1}\frac{U(l)U(k_1-l)U(k-k_1-1)}{k}, \qquad (3.9)$$

$U_0(0) = 0, \quad U_0(1) = 1, \quad U_0(2) = 0, \ldots$

For$U_1(k), we obtain$:

$U_1(0) = 0, \quad U_1(1) = 0, \quad U_1(2) = 0, \quad U_1(3) = 0, \quad U_1(4) = 0, \quad U_1(5) = \frac{1}{20}, \quad U_1(6) = 0, \quad U_1(7) = 0, \quad U_1(8) = 0, \quad U_1(9) = \frac{1}{1440},$

For$U_2(k), we obtain$:

$U_2(0) = 0, \quad U_2(1) = 0, \quad U_2(2) = 0, \quad U_2(3) = 0,$

$U_2(4) = 0, \quad U_2(5) = 0, \quad U_2(6) = 0, \quad U_2(7) = \frac{1}{840},$

Based on equation(11.2),we have the following:

$U(0) = 0, \quad U(1) = 1, \quad U(2) = 0, \quad U(3) = 0, \quad U(4) = 0,$

$U(5) = \frac{1}{20}, \quad U(6) = 0, \quad U(7) = \frac{1}{840}, \quad U(8) = 0, \quad U(9) = \frac{1}{1440},$

Therefore, the solution of the integral equation (6.3) is given by

$u(x) = x + \frac{1}{20}x^5 + \frac{1}{840}x^7 + \frac{1}{1440}x^9 + \cdots$

Example 3.4. Let us consider the following nonlinear Volterra integral equation:

$$u(x) = 1 + x + \int_0^x \int_0^x \int_0^x u^3(t)dt. \qquad (3.10)$$

Based on Theorem 2, we have the following:

$$u(x) = 1 + x + \frac{1}{2}\int_0^x (x-t)^2 u^3(t)dt,$$

Using differential of $u(x)$,we obtain:

$$\frac{d^2u(x)}{dx^2} = \int_0^x u^3(t)dt,$$

And using properties of DTM,

$$(k+1)(k+2)U(k+2) = \sum_{k_1=0}^{k-1}\sum_{l=0}^{k_1}\frac{U(l)U(k_1-l)U(k-k_1-1)}{k}, \qquad (3.11)$$

$U(0) = 1, \quad U(1) = 1, \quad U(2) = 0, \quad U(3) = \frac{1}{6}, \quad U(4) = \frac{1}{8},$

…

$u(x) = 1 + x + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots$

4. Conclusion

In this paper we could successfully use the techniques which make the solution of nonlinear and linear Volterra integral equation by differential transform method easy for us. It does not need the difficult calculation.

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