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ON SOME GENERALIZED FINSLER CONNECTION WITH TORSION

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Abstract
In the present article we generalize the connection of M. Hashiguchi and S. Hojo and study some uniqueness theorems and finally obtain the relationship between Cartan connection and the so-called Finsler connection.

Keywords: Generalized Finsler connection, Hashiguchi and Hojo connection, Cartan connection, Chern-Rund connection, Berwald connection.

AMS subject classification 53C60, 53B40.

1. Introduction.

There are four Finsler connections on a Finsler space: The connections named Berwald,Cartan, Chern-Rund and Hashiguchi connection, respectively. The theory of connections is an important field of differential geometry. It was initially developed to solve pure geometrical problems.
The most important linear connections in Finsler geometry were studied in ([1], [2], [3], [4], [5], ... etc.). Recently N. L. Youssef et al. [6] studied the Finslerian connections globally.
Historically in 1934 E. Cartan [7] established a connection which is metrical and torsion free. In 1970 M. Matsumbto [8] determined uniquely the Cartan connection by the following conditions:

1) Metrical,
2) Without torsion tensors and

3) With trivial deflection tensor i. e. with non-linear connection obtained from contracting F^i_{jk} by y^j.

On the other hand M. Hashiguchi [9] established the connection with the given torsion tensor T, while S. Hojo [10] established the new connection known as $\overset{(P)}{C}\Gamma$ in which the torsion tensor T vanished but it is not metrical with v-covariant derivative.

In the present article we have generalized the connection of M. Hashiguchi and S. Hojo. In process of determination of connection coefficient, we have taken the covariant derivatives of $\overset{(P)}{\Phi}_{ij}$ to vanish just as covariant derivatives of g_{ij} vanish in Cartan connection where Φ_{ij} is defined in the next sub-article.

2. The generalized Finsler Connection $\overset{(P)}{C}\Gamma T$

Let p≠1 be a real number, we define $\overset{(p)}{\Phi}(x,y)$ as

$$\overset{(p)}{\Phi} = \frac{1}{L}L^p \quad if \quad p \neq 0 \ , \ \overset{(p)}{\Phi} = \log L \quad if \qquad p = 0, \tag{2.1}$$

where L(x,y) is fundamental function. We write,

$$\dot{\partial}_i \overset{(P)}{\Phi} = \overset{(P)}{\Phi}_i, \qquad \dot{\partial}_i \dot{\partial}_j \overset{(P)}{\Phi} = \overset{(P)}{\Phi}_{ij} \qquad \text{and so on.}$$

Thus if l_i denote the unit vector along element of support then,

$$\text{a) } \overset{(P)}{\Phi}_i = L^{p-1} l_i, \qquad \text{b) } \overset{(P)}{\Phi}_{ij} = L^{p-2}\left[g_{ij} + (P-2) l_i l_j \ \right] \tag{2.2}$$

where $g_{ij}(x,y)$ is metric tensor.

Let us assume that the matrix of $\overset{(P)}{\Phi}_{ij}$ is non-singular. Then its inverse $\overset{(P)}{\Phi}_{ij}$ is given by,

$$\overset{(p)}{\Phi}{}^{ij} = L^{-(p-2)}[g^{ij} - \frac{p-2}{p-1} l^i l^j] \tag{2.3}$$

Differentiating (2.2)(b) by y^k we get

$$\overset{(P)}{\Phi}_{ijk} = L^{p-2}\left[2g_{ijk} + (P-2)L^{-1}\left(h_{ij} l_k + h_{jk} l_i + h_{ki} l_j + (P-1) l_i l_j l_k\right)\right], \tag{2.4}$$

where h_{ij} is the angular metric tensor defined as $h_{ij}=g_{ij}-l_i l_j$.

Definition 2.1. A Finsler connection FΓ= $\left(F^i_{jk}, N^i_j, C^i_{jk}\ \right)$ on Finsler manifold M^n defined as a traid consisting of a v-connection F^i_{jk}, a non-linear connection N^i_j and a vertical connection C^i_{jk} and satisfying the following axioms is called the Cartan connection and is denoted by CΓ,

A_1) $g_{ij|k} = 0$ $\qquad$ A_2) (h)h-torsion $T^i_{jk} = 0$ $\qquad$ A_3) deflection tensor $D^i_j = 0$

A4) g_{ij} |k=0 A5) (v)v-torsion $S^{i}_{jk} = 0$,

where "|" and "|" are the symbols for h-and v-convariant differentioation respectively.
To distinguish a general Finsler connection and Cartan's connection we shall denote Cartan connection CΓ as ($\Gamma^{*i}_{jk}, N^{i}_{j}, C^{i}_{jk}$), where

$$\Gamma^{*i}_{jk} = \gamma^{i}_{jk} - C^{i}_{km} N^{m}_{j} - C^{i}_{jm} N^{m}_{k} + g^{hi} C_{jkm} N^{m}_{h} \tag{2.5}$$

and

$$N^{i}_{j} = \gamma^{i}_{jk} y^{k} - g^{i}_{jm} \gamma^{m}_{hl} y^{h} y^{l}, \tag{2.6}$$

where $\gamma^{h}_{jk} = g^{hj} \gamma_{ijk}$ is the chrotoffel's symbol of second kind and $\gamma_{ijk} = \frac{1}{2}\{\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\}$ is the christoffel's symbol of first kind.

Theorem 2.1. The generalized Finsler connection $(\overset{(P)}{F}{}^{i}_{jk}, \overset{(p)}{N}{}^{i}_{j}, \overset{(p)}{C}{}^{i}_{jk})$ is uniquely determined by

$g_{ij||k}=0$ (connection is h-metrical), (2.7)

$$\overset{(p)}{N}{}^{r}_{j} = y^{r} \overset{(p)}{F}{}^{i}_{rj} \quad \text{(deflection tensor field vanishes)}, \tag{2.8}$$

(h) h- torsion tensor $\overset{(p)}{T}{}^{i}_{jk} = \overset{(p)}{F}{}^{i}_{jk} - \overset{(p)}{F}{}^{i}_{kj}$ (2.9)

is a given skew symmetric and (0) p homogeneous tensor

$$\overset{(p)}{\Phi}_{ij}\|_{k} = 0 \ , \quad \overset{(p)}{C}{}^{i}_{jk} = \overset{(p)}{C}{}^{i}_{kj} \ , \tag{2.10}$$

where ||k and ||k denote h and v- covariant derivative with respect to the connection $\overset{(P)}{C\Gamma}$.
Proof. From (2.7) it follows that

$$g_{ij||k} = \partial_k g_{ij} - g_{rj} \overset{(P)}{F}{}^{r}_{ik} - g_{ir} \overset{(p)}{F}{}^{r}_{jk} = 0 \tag{2.7)'$$

Applying Christoffel process to (2.7)' we get

$$\overset{(p)}{F}{}^{i}_{jk} = \gamma^{i}_{jk} - \overset{(p)}{N}{}^{m}_{k} g^{i}_{mj} - \overset{(p)}{N}{}^{m}_{j} g^{i}_{mk} + g^{ri} \overset{(p)}{N}{}^{m}_{r} g_{mjk} - \frac{1}{2}(\overset{(P)}{T}{}^{i}_{kj} + g_{km} g^{ir} \overset{(p)}{T}{}^{m}_{rj} + g_{mj} g^{ir} \overset{(p)}{T}{}^{m}_{rk}). \tag{2.11}$$

Contracting (2.11) with y^i and using (2.8) we get

$$\overset{(p)}{N}{}^{i}_{k} = \gamma^{i}_{0k} + \overset{(p)}{N}{}^{m}_{0} g^{i}_{mk} - \frac{1}{2}(\overset{(p)}{T}{}^{i}_{k0} + g_{km} g^{ir} \overset{(p)}{T}{}^{m}_{r0} + g^{ir} \overset{(p)}{T}{}^{0}_{rk}), \tag{2.12}$$

where o denotes the contraction with y^i i. e. $\gamma^{i}_{ok} = \gamma^{i}_{jk} y^{j}$.

Again contracting (2.12) with y^k we get

$$\overset{(p)}{N}{}^{i}_{o} = \gamma^{i}_{oo} - g^{ri}\,\overset{(p)}{T}{}^{o}_{ro}\,. \qquad (2.\,13)$$

Substituting (2. 13) in (2.12) we get

$$\overset{(P)}{N} = \gamma^{i}_{ok} - (\gamma^{m}_{ok} - g^{rm}\,\overset{(p)}{T}{}^{o}_{ro})g^{i}_{mk} - \frac{1}{2}(\overset{(p)}{T}{}^{i}_{ko} + g_{km}g^{ir}\,\overset{(p)}{T}{}^{m}_{ro} + g^{ir}\,\overset{(p)}{T}{}^{o}_{rk}) \qquad (2.\,14)$$

From (2.11) and (2. 14) we get unique $\overset{(p)}{F}{}^{i}_{jk}$ and $\overset{(p)}{N}{}^{i}_{j}$ with given $\overset{(p)}{T}{}^{i}_{jk}$.

Form (2. 10) we get

$$\overset{(P)}{\Phi}_{ij}\Big\|_{k} = \overset{(P)}{\Phi}_{ijk} - \overset{(P)}{C}_{ijk} - \overset{(P)}{C}_{jik} = 0, \qquad (2.15)$$

where $\overset{(P)}{C}_{ijk} = \overset{(P)}{\Phi}_{rj}\,\overset{(p)}{C}{}^{r}_{ik}$. Applying Christoffel process to (2.15) and using the axiom

$\overset{(p)}{C}{}^{i}_{jk} = \overset{(p)}{C}{}^{i}_{kj}$ we get,

$$\overset{(P)}{C}_{ijk} = \frac{1}{2}\left[\overset{(P)}{\Phi}_{ijk} + \overset{(P)}{\Phi}_{jki} - \overset{(P)}{\Phi}_{kij}\right] = \frac{1}{2}\overset{(P)}{\Phi}_{ijk}\,, \qquad (2.16)$$

which gives

$$\overset{(p)}{C}{}^{r}_{ik} = \frac{1}{2}\overset{(p)}{\Phi}{}^{rj}\,\overset{(p)}{\Phi}_{ijk} = g^{r}_{ik} + \overset{(p)}{\sigma}{}^{r}_{ik}\,, \qquad (2.17)$$

where $\overset{(p)}{\sigma}{}^{r}_{ik}$ is given by

$$\overset{(p)}{\sigma}{}^{r}_{ik} = \frac{p-2}{2L}\left\{\delta^{r}_{i}l_{k} + \delta^{r}_{k}l_{i} + \frac{h_{ik}l^{r}}{(p-1)} - l_{i}l_{k}l^{r}\right\}.$$

Thus from (2.17) we get unique $\overset{(p)}{C}{}^{r}_{ik}$.

Theorem 2.2. Any connection ($(\overset{(p)}{F}{}^{i}_{jk}, \overset{(p)}{N}{}^{i}_{j}, \overset{(p)}{C}{}^{i}_{jk})$) determined by theorem (2.1) also satisfy $\overset{(P)}{\Phi}_{ij\|k} = 0$ for p≠0.

Proof. Conditions (2.7) and (2.8) of theorem (2.1) are equivalent to $g_{ij\|k} = 0, y^{i}_{\|k} = 0$, which in view of relation $L^2 = g_{ij}y^{i}y^{j}$ gives $L_{\|k}=0$. Hence $l_{i\|k} = 0$, which in view of (2.2) yields

$$\overset{(P)}{\Phi}_{ij\|k} = 0\,.$$

Theorem 2.3. The generalized Finsler connection ($(\overset{(p)}{F}{}^{i}_{jk}, \overset{(p)}{N}{}^{i}_{j}, \overset{(p)}{C}{}^{i}_{jk})$) is uniquely determined for p≠0 by

$$\overset{(P)}{\Phi}_{ij\|k} = 0\,, \qquad \overset{(P)}{\Phi}_{ij}\Big\|_{k} = 0 \qquad (2.18)$$

$$\overset{(p)}{N}{}^{i}_{j} = y^{r}\,\overset{(p)}{F}{}^{i}_{rj} \qquad (2.19)$$

(h)h – torsion tensor $\overset{(P)}{T}{}^{i}_{jk}$ is given skew symmetric and (0)p homogeneous tensor (2.20)

$$\overset{(P)}{C}{}^{i}_{jk} = \overset{(P)}{C}{}^{i}_{kj}\,, \qquad (2.21)$$

and is the same connection as that determined in theorem (2.1)

Proof. If we take the connection $(\overset{(p)}{F}{}^i_{jk}, \overset{(p)}{N}{}^i_j, \overset{(p)}{C}{}^i_{jk})$ determined by theorem (2.1), then theorem (2.2) gives $\overset{(P)}{\Phi}_{ij\|k} = 0$. If we take the connection determined by axioms (2.18) to (2.21), then from (2.19) we get $\overset{(p)}{y}{}^i_{\|k} = 0$. Also from homogeneity of $\overset{(p)}{\Phi}_i$ we have

$$\overset{(P)}{\Phi}_{ij} y^i = (P-1)\overset{(P)}{\Phi}_i ,$$

which gives $\overset{(P)}{\Phi}_{i\|k} = 0$. Again $\overset{(P)}{\Phi}_i y^i = \overset{(P)}{\Phi}$ implies that $\overset{(P)}{\Phi}_{\|k} = 0$. Hence from (2.1) we have $L_{\|k} = 0$, from which we get $l_{i\|k} = 0$.

Thus equation (2.2) gives $g_{ij\|k} = 0$. The uniqueness follows from theorem (2.1).

3. relation between Cartan connection and the generalized Finsler Connection $\overset{(P)}{C\Gamma T}$.

We establish the relation between Cartan connection CΓ=($\Gamma^{*i}_{jk}, N^i_j, C^i_{jk}$) and the generalized Finsler connection $\overset{(P)}{C\Gamma T} = (\overset{(p)}{F}{}^i_{jk}, \overset{(p)}{N}{}^i_j, \overset{(p)}{C}{}^i_{jk})$.

In view of (2.11), (2.14), (2.5) and (2.6) we have the following relations

$$\overset{(p)}{F}{}^i_{jk} = \Gamma^{*i}_{jk} + A^i_{jk}, \tag{3.1}$$

$$\overset{(p)}{N}{}^i_j = N^i_j + A^i_{oj}, \tag{3.2}$$

where

$$\begin{aligned} A^i_{jk} &= (g^{is}_m g^m_{jk} - g^i_{mj} g^{ms}_k - g^i_{mk} g^{ms}_j)\overset{(p)}{T}{}^o_{so} \\ &+ \frac{1}{2}(g^i_{mj}\overset{(p)}{T}{}^m_{ko} + g^i_{mk}\overset{(p)}{T}{}^m_{jo}\, g^{ri} g_{mjk}\overset{(p)}{T}{}^m_{ro}) + \frac{1}{2}(g^{is}_j \overset{(P)}{T}{}_{sko} + g^{is}_k \overset{(P)}{T}{}_{sjo} - g^s_{jk}\overset{(p)}{T}{}^i_{so}) \\ &+ \frac{1}{2}(g^{ir}_j \overset{(p)}{T}{}^o_{rk} + g^{ir}_k \overset{(p)}{T}{}^o_{rj} - g^s_{jk} g^{ri}\overset{(p)}{T}{}^o_{sr}) - \frac{1}{2}(\overset{(p)}{T}{}^i_{kj} + g_{km} g^{ir}\overset{(p)}{T}{}^m_{rj} + g_{mj} g^{ir}\overset{(p)}{T}{}^m_{rk}) \end{aligned} \tag{3.3}$$

And

$$\overset{(P)}{T}{}_{sko} = g_{kr}\overset{(p)}{T}{}^r_{so}. \tag{3.4}$$

Now if $U^i_{j\|k}$ and $U^i_{j|k}$ denote h-covariant derivative of a tensor field U^i_j with respect to the connection $\overset{(P)}{C\Gamma T}$ and Cartan connection CΓ respectively, then in view of (2.17)

$$U^i_j\|_k = U^i_j|_k + U^r_j\overset{(p)}{\sigma}{}^i_{rk} - U^i_r\overset{(P)}{\sigma}{}^r_{jk} \tag{3.5}$$

The (v)h – torsion tensor and (v)hv – torsion tensor of $\overset{(P)}{C\Gamma T}$ are given by

$$\overset{(p)}{R}{}^i_{jk} = \delta_k \overset{(p)}{N}{}^i_j - \delta_j \overset{(p)}{N}{}^i_k, \tag{3.6}$$

$$\overset{(P)}{P}{}^i_{jk} = \dot{\partial}_k \overset{(p)}{N}{}^i_j - \overset{(p)}{F}{}^i_{kj} , \qquad (3.7)$$

where $\partial_k = \partial_k - N^r_k \dot{\partial}_r$

In view of (3.1) and (3.2) we have

$$\overset{(p)}{R}{}^i_{jk} = R^i_{jk} + A^i_{oj|k} - A^i_{ok|j} - F^i_{ojr} A^r_{ok} + F^i_{okr} A^r_{oj} - (\dot{\partial}_r A^i_{oj}) A^r_{ok} + (\dot{\partial}_r A^r_{ok}) A^r_{oj} , \qquad (3.8)$$

$$\overset{(p)}{P}{}^i_{jk} = P^i_{jk} + A^i_{ojk} , \qquad (3.9)$$

where R^i_{jk} and P^i_{jk} are (v)h-torsion tensor and (v)hv – torsion tensor of CΓ respectively and

$$F^i_{hjr} = \dot{\partial}_r \Gamma^{*i}_{hj} , \qquad A^i_{mjk} = \dot{\partial} A^i_{mj} . \qquad (3.10)$$

Here F^i_{hjr} is the hv- curvature tensor of Chern-Rund connection.

References

[1] Bao, D. and Chern, S.S. On a notable connection in Finsler geometry, Houston J. Math 19(1993), 130-180.

[2] Bao, D. & Chern, s. s. and Shen, Z.: An introduction to Riemann-Finsler geometry, springer-verlag, Berlin, (2000).

[3] Matsumoto, M.: Foundation of Finsler geometry and special Finsler space, Kaiseisha press, Otsu, 520 Japan, (1986).

[4] Miron, R. and Anastasiei: The geometry of Lagrange spaces; theory and application, Kluwer Acad publ. No 59, (1994).

[5] Tamassy, L.: Area and metrical connections in Finsler space. Finsler geometry, A Meeting of Minds, Kluwer Publ. (2000), proc. of Coll. on Finsler spaces, Edmonton, Jully (1998), 263-280.

[6] Youssef, N. L.; Abed, S. H. and Soleiman, A.: A global approach to the theory of connections in Finsler geometry: arxiv. org (2008).

[7] Cartan, E.: Les espuces de Finsler. Actualites 79, Paris (1934) 41 P, 2nd edit. (1971).

[8] Matsumoto, M.: The theory of Finsler Connections. Publ. of the study group of geometry 5, Deptt. Math. Okayama Univ. (1970), 120 p.

[9] Hashinguchi, M.: On determination of Finsler connection by deflection tensor fields. Rep. Fac. Dei. Kageshima univ. Math. Phys. Chem. 2 (1969) 29-39.

[10] Hojo, S.: On determination of generalized Cartan connection and fundamental functions of Finsler spaces, Tensor, N. S. 35 (1981), 333-344.

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