



BPs Operational Matrices for Solving Time Varying Fractional Optimal Control Problems

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Abstract

In this paper, we present a method for solving time varying fractional optimal control problems by Bernstein polynomials. Firstly, we derive the Bernstein polynomials (BPs) operational matrix for the fractional derivative in the Caputo sense, which has not been undertaken before. This method reduces the problems to a system of algebraic equations. The results obtained are in good agreement with the existing ones in open literatures and the solutions approach to classical solutions as the order of the fractional derivatives approach to 1.

Keywords: Time varying fractional optimal control problems, Bernstein polynomials, operational matrix, Caputo derivative.

1. Introduction

Although the optimal control theory is an area in mathematics which has been under development for years but the fractional optimal control theory is a very new area in mathematics. Recent contributions in this field were reported by several authors [1-4].

In this paper, we consider the time varying fractional optimal control problem as follows:

$$\text{Minimize } J(x(t), u(t)) = \frac{1}{2} \int_0^1 x^2(t) + u^2(t) dt, \quad (1)$$

subject to the dynamic constrain

$$D^\alpha x(t) = a_1(t)x(t) + a_2(t)u(t), \quad 0 < t \leq 1, \quad 0 < \alpha \leq 1, \tag{2}$$

and the initial condition

$$x(0) = x_0, \tag{3}$$

where $x(t)$ and $u(t)$ are the state function and the control function, respectively. When $\alpha = 1$, the above problem reduces to a standard optimal control problem.

The rest of this paper is as follows. In Section 2, we present some preliminaries in fractional calculus. In Section 3, BPs are introduced and then we approximate functions by using BPs and we show the properties of BPs by several Lemmas and corollaries. We make a new operational matrix for fractional derivative by BPs in Section 4. In Section 5, we apply BPs for solving time varying fractional optimal control problems. In Section 6, numerical examples are simulated to demonstrate the high performance of the proposed method. Finally, Section 7 concludes our work in this paper.

2. Some preliminaries in fractional calculus

In this section, we give some basic definitions and properties of the fractional calculus which are used further in this paper.

Definition 2.1. (See [5]) We define

$$C_\mu = \left\{ f(t) \mid f(t) > 0 \text{ for } t > 0 \text{ and } f(t) = t^p f_1(t) \text{ where } p > \mu \text{ and } f_1(t) \in C[0, \infty) \right\}, \text{ and}$$

$$C_\mu^n = \left\{ f(t) \mid f^{(n)}(t) \in C_\mu \right\} \text{ where } n \in \mathbb{N}, \mu \in \mathbb{R}.$$

Definition 2.2. (See [5]) The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \quad \alpha > 0, t > 0, \tag{4}$$

$$I^0 f(t) = f(t),$$

and for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $f \in C_{-1}^n$, the fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x) dx. \tag{5}$$

Property 2.3. (See [6-8]) For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ we have

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \tag{6}$$

and for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $f \in C_\mu^n$, $\mu \geq -1$ we see the following properties

$$1. D^\alpha I^\alpha f(t) = f(t), \tag{7}$$

$$2. I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad t > 0, \tag{8}$$

$$3. D^\beta f(t) = I^{\alpha-\beta} D^\alpha f(t). \tag{9}$$

3. Bernstein polynomials and their properties

The Bernstein polynomials (BPs) of m th-degree are defined on the interval $[0,1]$ as follows:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \dots, m. \tag{10}$$

Corollary 3.1. Set $\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$ is a complete basis in Hilbert space $L^2[0,1]$ and polynomials of degree m can be expanded in terms of linear combination of $B_{i,m}(x)$ ($i = 0, 1, \dots, m$) as follows

$$P(x) = \sum_{i=0}^m c_i B_{i,m}(x). \tag{11}$$

Lemma 3.2. We can write $\Phi_m(x) = A T_m(x)$, where A is a matrix upper triangular, $T_m(x) = [1, x, \dots, x^m]^T$ and $\Phi_m(x) = [B_0(x), B_2(x), \dots, B_m(x)]^T$.

Proof. Using binomial expansion of $(1-x)^{m-i}$, we have

$$\begin{aligned} B_{i,m}(x) &= \binom{m}{i} x^i (1-x)^{m-i} = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^k \right) \\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k}, \quad i = 0, 1, \dots, m. \end{aligned}$$

Therefore we can write

$$\Phi_m(x) = A T_m(x), \tag{12}$$

$$\text{where } A = (a_{i,j})_{i,j=1}^{m+1} \text{ and } a_{i+1,j+1} = \begin{cases} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} & i \leq j, \\ 0 & i > j, \end{cases} \quad i, j = 0, 1, \dots, m. \quad \square$$

Lemma 3.3. (see [9]) Let $L^2[0,1]$ be a Hilbert space with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$, and $y \in L^2[0,1]$. Then, we can find the unique vector $c = [c_1, c_2, \dots, c_m]^T$ such that

$$y(x) \approx \sum_{i=0}^m c_i B_{i,m}(x) = c^T \Phi_m(x). \tag{13}$$

Corollary 3.4. In lemma 3.3, we have $c^T = \langle f, \Phi_m \rangle Q^{-1}$, such that

$\langle f, \Phi_m \rangle = \int_0^1 f(x) \Phi_m(x)^T dx = [\langle f, B_{0,m} \rangle, \langle f, B_{1,m} \rangle, \dots, \langle f, B_{m,m} \rangle]$ and $Q = (Q_{i,j})_{i,j=1}^{m+1}$ is as follows

$$Q_{i+1,j+1} = \int_0^1 B_{i,m}(x) B_{j,m}(x) dx = \frac{\binom{m}{i} \binom{m}{j}}{(2m+1) \binom{2m}{i+j}}, \quad i, j = 0, 1, \dots, m. \tag{14}$$

Lemma 3.5. Suppose that $c_{(m+1) \times 1}$ is an arbitrary vector. The operational matrix of product $\hat{C}_{(m+1) \times (m+1)}$ using BPs can be given as follows:

$$c^T \Phi_m(x) \Phi_m(x)^T \approx \Phi_m(x)^T \hat{C}. \tag{15}$$

Proof. From (12) we have

$$\begin{aligned} c^T \Phi_m(x) \Phi_m(x)^T &= c^T \Phi_m(x) (T_m(x)^T A^T) = [c^T \Phi_m(x), x(c^T \Phi_m(x)), \dots, x^m(c^T \Phi_m(x))] A^T \\ &= \left[\sum_{i=0}^m c_i B_{i,m}(x), \sum_{i=0}^m c_i x B_{i,m}(x), \dots, \sum_{i=0}^m c_i x^m B_{i,m}(x) \right] A^T. \end{aligned}$$

Now, we approximate all functions $x^k B_{i,m}(x)$ in terms of $\Phi_m(x)$. Thus we define

$e_{k,i} = [e_{k,i}^0, e_{k,i}^1, \dots, e_{k,i}^m]^T$, then by (14) we can write

$$x^k B_{i,m}(x) \approx e_{k,i} \Phi_m(x), \quad i, k = 0, 1, \dots, m.$$

So, we get

$$\begin{aligned}
 e_{k,i} &= Q^{-1} \left(\int_0^1 x^k B_{i,m}(x) \Phi_m(x) dx \right) \\
 &= Q^{-1} \left[\int_0^1 x^k B_{i,m}(x) B_{0,m}(x) dx, \int_0^1 x^k B_{i,m}(x) B_{1,m}(x) dx, \dots, \int_0^1 x^k B_{i,m}(x) B_{m,m}(x) dx \right]^T \\
 &= \frac{Q^{-1} \binom{m}{i}}{2m+k+1} \left[\frac{\binom{m}{0}}{\binom{2m+k}{i+k}}, \frac{\binom{m}{1}}{\binom{2m+k}{i+k+1}}, \dots, \frac{\binom{m}{m}}{\binom{2m+k}{i+k+m}} \right]^T, \quad i, k = 0, 1, \dots, m.
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 \sum_{i=0}^m c_i x^k B_{i,m}(x) &\approx \sum_{i=0}^m c_i \left(\sum_{j=0}^m e_{k,i}^j B_{j,m}(x) \right) = \sum_{i=0}^m B_{j,m}(x) \left(\sum_{i=0}^m c_i e_{k,i}^j \right) \\
 &= \Phi_m(x)^T \left[\sum_{i=0}^m c_i e_{k,i}^0, \sum_{i=0}^m c_i e_{k,i}^1, \dots, \sum_{i=0}^m c_i e_{k,i}^m \right]^T \\
 &= \Phi_m(x)^T [e_{k,0}, e_{k,1}, \dots, e_{k,m}] c = \Phi_m(x)^T V_{k+1} c,
 \end{aligned}$$

where $V_{k+1} (k = 0, 1, \dots, m)$ is an $(m+1) \times (m+1)$ matrix that has vectors $e_{k,i} (i = 0, 1, \dots, m)$ for each column's. If we define $\bar{C} = [V_1 c, V_2 c, \dots, V_m c]$, then we get

$$c^T \Phi_m(x) \Phi_m(x)^T \approx \Phi_m(x)^T \bar{C} A^T, \tag{16}$$

and therefore we obtain the operational matrix of product $\hat{C} = \bar{C} A^T$.

4. BPs operational matrix for fractional derivative

In this section, we obtain the operational matrix for the fractional derivative. We can write

$$D^\alpha \Phi_m(t) = \frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1} * \Phi_m^{(n)}(t), \quad 0 \leq t \leq 1, \tag{17}$$

where $*$ denotes the convolution product.

By (12) we obtain

$$D^\alpha \Phi_m(t) = \frac{1}{\Gamma(n-\alpha)} A \left(t^{n-\alpha-1} * T_m^{(n)}(t) \right) = A D^\alpha T_m(t) = A [D^\alpha 1, D^\alpha t, \dots, D^\alpha t^m]^T, \tag{18}$$

where $D^\alpha t^j = \begin{cases} 0 & j = 0, \dots, [\alpha] - 1, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} t^{j-\alpha} & j = [\alpha], \dots, m. \end{cases}$

Therefore we have

$$D^\alpha T_m(t) = \tilde{\Sigma} \tilde{T}, \tag{19}$$

where $\tilde{\Sigma}$ and \tilde{T} are a $(m+1) \times (m+1)$ diagonal matrix and a $(m+1) \times 1$ matrix, respectively as follows:

$$\tilde{\Sigma} = (\tilde{\Sigma}_{i,j})_{i,j=1}^{m+1}, \quad \tilde{\Sigma}_{i+1,j+1} = \begin{cases} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} & i, j = [\alpha], \dots, m \text{ and } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

and $\tilde{T} = (\tilde{T}_i) = \begin{cases} 0 & i = 0, \dots, [\alpha] - 1, \\ t^{i-\alpha} & i = [\alpha], \dots, m. \end{cases}$ \tag{21}

Now, we approximate $t^{i-\alpha}$ ($i = [\alpha], \dots, m$) with respect to BPs by using (13). Therefore, we can write

$$t^{i-\alpha} \approx P_i^T \Phi_m(t), \tag{22}$$

where P_i , ($i = [\alpha], \dots, m$) is a vector $(m+1) \times 1$. So, we have

$$P_i = Q^{-1} \left(\int_0^1 t^{i-\alpha} \Phi_m(t) dt \right) = Q^{-1} \left[\int_0^1 t^{i-\alpha} B_{0,m}(t) dt, \int_0^1 t^{i-\alpha} B_{1,m}(t) dt, \dots, \int_0^1 t^{i-\alpha} B_{m,m}(t) dt \right]^T \\ = Q^{-1} \bar{P}_i,$$

where

$$\bar{P}_i = [\bar{P}_{i,0}, \bar{P}_{i,1}, \dots, \bar{P}_{i,m}]^T, \tag{23}$$

$$\bar{P}_{i,j} = \int_0^1 t^{i-\alpha} B_{j,m}(t) dt = \frac{m! \Gamma(i+j-\alpha+1)}{j! \Gamma(i+m-\alpha+2)}, \quad i = [\alpha], \dots, m \text{ and } j = 0, 1, \dots, m. \tag{24}$$

Now, we suppose P is an $(m+1) \times (m+1)$ matrix that has vector zero in $[\alpha]$ first column and vector P_i in $(i+1)$ th column's for $i = [\alpha], \dots, m$.

Finally, from (17)-(24), we obtain

$$D^\alpha \Phi_m(t) \approx D_\alpha \Phi_m(t), \tag{25}$$

where

$$D_\alpha \approx A \tilde{\Sigma} P^T, \tag{26}$$

is called the Bernstein polynomials operational matrix of fractional derivative.

6. BPs for solving time varying fractional optimal control problems

Using Lemma 3.3, we can approximate the functions as follows:

$$x(t) \approx c^T \Phi_m(t), \tag{27}$$

$$u(t) \approx b^T \Phi_m(t), \tag{28}$$

$$a_1(t) \approx a_1^T \Phi_m(t), \tag{29}$$

$$a_2(t) \approx a_2^T \Phi_m(t), \tag{30}$$

where $c, b, a_1, a_2 \in R^{(m+1) \times 1}$.

by (25) and (27) we can write

$$D^\alpha x(t) \approx c^T D_\alpha \Phi_m(t). \tag{31}$$

Therefore, the problem (1)-(3) reduce to the following problem:

$$\text{Minimize } \frac{1}{2} \int_0^1 c^T \Phi_m(t) \Phi_m(t)^T c + b^T \Phi_m(t) \Phi_m(t)^T b dt, \tag{32}$$

subject to the dynamic constrain

$$c^T D_\alpha \Phi_m(t) = a_1^T \Phi_m(t) \Phi_m(t)^T c + a_2^T \Phi_m(t) \Phi_m(t)^T b, \tag{33}$$

and the initial condition

$$c^T \Phi_m(0) = x_0. \tag{34}$$

Now, using Corollary 3.4 for (32) we can write

$$\begin{aligned} \text{Minimize } J(c,b) &= \frac{1}{2}c^T \left(\int_0^1 \Phi_m(t)\Phi_m(t)^T dt \right) c + \frac{1}{2}b^T \left(\int_0^1 \Phi_m(t)\Phi_m(t)^T dt \right) b \\ &= \frac{1}{2}c^T Qc + \frac{1}{2}b^T Qb \end{aligned} \tag{35}$$

Also by Lemma 3.5 for (33) we have

$$c^T D_\alpha \Phi_m(t) = \Phi_m(t)^T \hat{A}_1 c + \Phi_m(t)^T \hat{A}_2 b. \tag{36}$$

Now, by using tau method [10] we can generate algebraic equations from (36) as follows

$$G_j(c,b) = \int_0^1 (c^T D_\alpha - c^T \hat{A}_1^T - b^T \hat{A}_2^T) \Phi_m(t) B_{j,m}(t) dt = 0, \quad j = 0, \dots, m-1, \tag{37}$$

and from (34) we set $G_m = c^T \Phi_m(0) - x_0$.

Finally, the problem (1)-(3) has been reduced to a parameter optimization problem which can be stated as follows:

Find c and b which

$$\text{Minimize } J(c,b) = \frac{1}{2}c^T Qc + \frac{1}{2}b^T Qb, \tag{38}$$

subject to the system of algebraic equations

$$G_j(c,b) = 0, \quad j = 0, \dots, m. \tag{39}$$

For solving the above problem we use the Lagrange multipliers method. So, we define Lagrange function for the problem (38) and (39) as follows:

$$L(c,b,\lambda) = \frac{1}{2}c^T Qc + \frac{1}{2}b^T Qb + \sum_{j=0}^m \lambda_j G_j(c,b), \tag{40}$$

where $\lambda = [\lambda_0, \dots, \lambda_m]^T$ is the unknown Lagrange multiplier. Now, we consider the necessary conditions for the extremum and obtain the following systems of algebraic equations

$$\frac{\partial L}{\partial c} = 0, \tag{41}$$

$$\frac{\partial L}{\partial b} = 0, \tag{42}$$

$$\frac{\partial L}{\partial \lambda} = 0. \tag{43}$$

Equations (41)-(43) can be solved for c, b and λ by Newton's iterative method. Then, we get the approximate value of the state functions $x(t)$ and the control functions $u(t)$ from (27) and (28), respectively.

7. Numerical examples

To demonstrate the applicability and to validate the numerical scheme, we apply the present method for the following examples.

Example 1. Consider the following time invariant problem [1, 2]

$$\text{Minimize } J = \frac{1}{2} \int_0^1 x^2(t) + u^2(t) dt,$$

subject to the system dynamics

$$D^\alpha x(t) = -x(t) + u(t),$$

with initial condition

$$x(0) = 1.$$

For this problem we have the exact solution in the case of $\alpha = 1$ as follows

$$\begin{aligned} x(t) &= \cosh(\sqrt{2} t) + \beta \sinh(\sqrt{2} t), \\ u(t) &= (1 + \sqrt{2} \beta) \cosh(\sqrt{2} t) + (\sqrt{2} + \beta) \sinh(\sqrt{2} t), \end{aligned}$$

$$\text{where } \beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2} \sinh(\sqrt{2})}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})}.$$

Figures 1 and 2 show the state and the control variables, respectively, as a function of time for $m = 5$, for different values of α . These figures show that as α approaches close to 1, the numerical solutions for both the state and the control variables approach to the analytical solutions for $\alpha = 1$ as expected. In Figures 1 and 2, only the numerical results for $\alpha = 1$ are presented. This is because, for $\alpha = 1$ the analytical and the numerical results overlap. In Figs. 3 and 4, we see the absolute error of obtained results for $m = 10$ and $\alpha = 1$. Also, in Table 1 and 2, the absolute error of $x(t)$ for when $\alpha = 1$ and $m = 4, 5$ is demonstrated and is compared with [1].

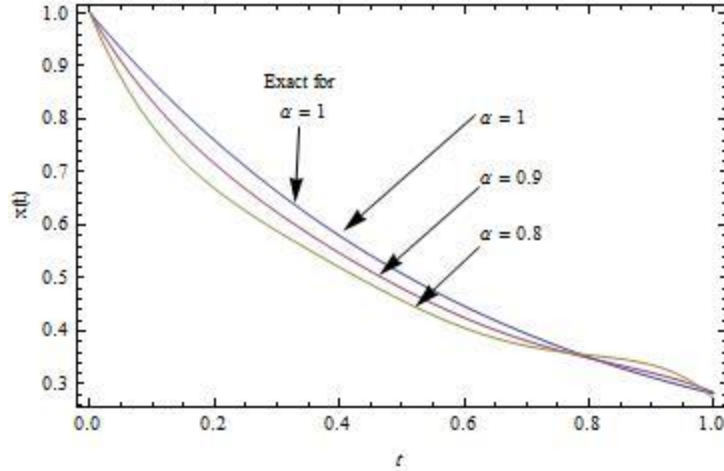


Fig 1. Approximate solutions of $x(t)$ for $m = 5$ and different values of α and exact solution for $\alpha = 1$ in example 1.

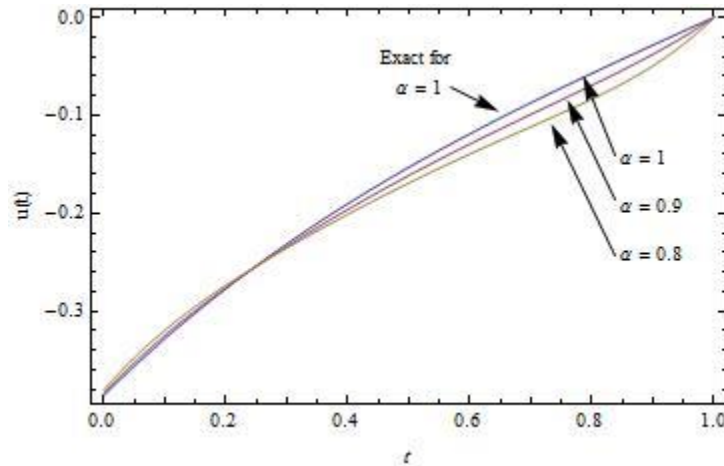


Fig 2. Approximate solutions of $u(t)$ for $m = 5$ and different values of α and exact solution for $\alpha = 1$ in example 1.

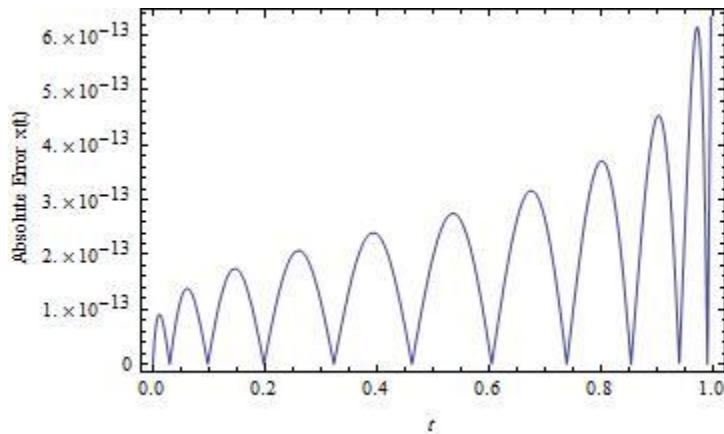


Fig 3. Plot of absolute error function $x(t)$ for $\alpha = 1$ and $m = 10$ in example 1.

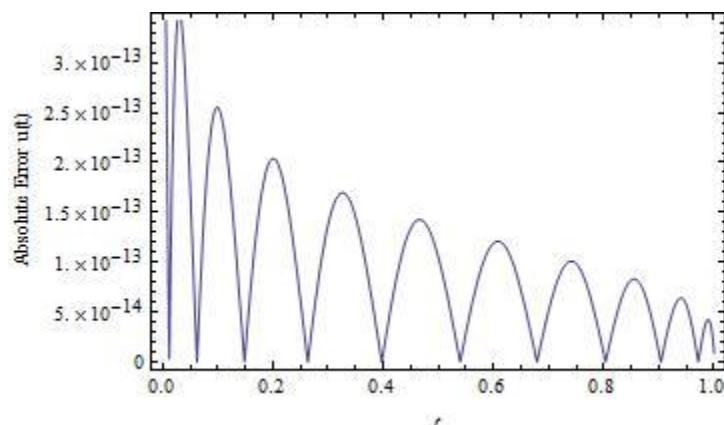


Fig 4. Plot of absolute error function $u(t)$ for $\alpha = 1$ and $m = 10$ in example 1.

Table 1. Absolute error $x(t)$ for $\alpha = 1$, $m = 4$ and different values t in example 1.

t	$m = 4$ [1]	Present Method $m = 4$
0	0.0000899	0
0.1	0.0000477	0.00001560360486785317
0.2	0.0000325	0.000026407578959108946
0.3	0.0000774	0.000036101485012540735
0.4	0.0000213	$3.899031830290056 \times 10^{-6}$
0.5	0.0000643	0.00003810091250144865
0.6	0.000103	0.00004946576317460538
0.7	0.000112	0.000010110645709493493
0.8	0.0000914	0.000055108397533421005
0.9	0.0000941	0.00005539673693399916

Table 2. Absolute error $x(t)$ for $\alpha = 1$, $m = 5$ and different values t in example 1.

t	[1] $m = 5$	Present Method $m = 5$
0	0.00000625	0
0.1	0.0000134	$1.2376247904288817 \times 10^{-7}$
0.2	0.0000212	$2.1260006768386575 \times 10^{-6}$
0.3	0.0000324	$1.787127701380342 \times 10^{-8}$
0.4	0.0000473	$2.6023598275193294 \times 10^{-6}$
0.5	0.0000620	$1.8611311257688001 \times 10^{-6}$
0.6	0.0000749	$1.6945524689004898 \times 10^{-6}$
0.7	0.0000888	$3.600201665232561 \times 10^{-6}$
0.8	0.000107	$1.4930994740591785 \times 10^{-7}$
0.9	0.000131	$5.091619992403373 \times 10^{-6}$

Example 2. This example considers a time varying fractional optimal control problem [1, 2]. Find the control $u(t)$ which minimizes the performance index J given in Example 1 subject to the following dynamical system

$$D^\alpha x(t) = t x(t) + u(t),$$

with initial condition

$$x(0) = 1.$$

Figs. 5 and 6 show the state $x(t)$ and the control $u(t)$ as functions of t for different values of α . These figures show that as α approaches close to 1, the numerical solutions for both the state and the control variables approach to the solutions for $\alpha=1$ as expected. The numerical solution obtained with the proposed method for fractional orders of α matches those found in the literature.

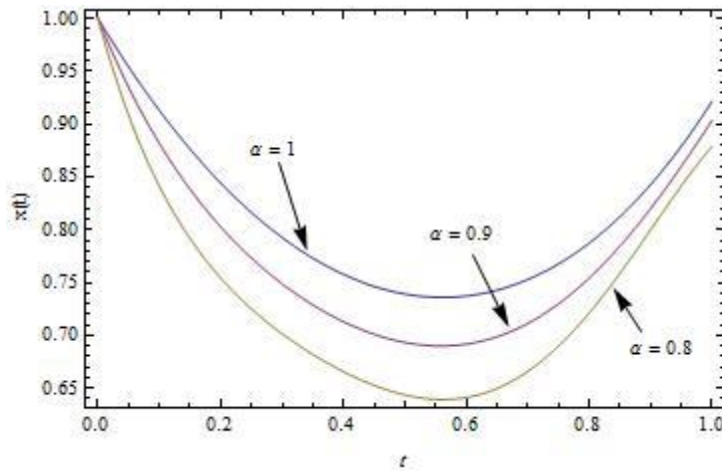


Fig 5. Approximate solutions of $x(t)$ for $m = 5$ and different values of α in example 2.

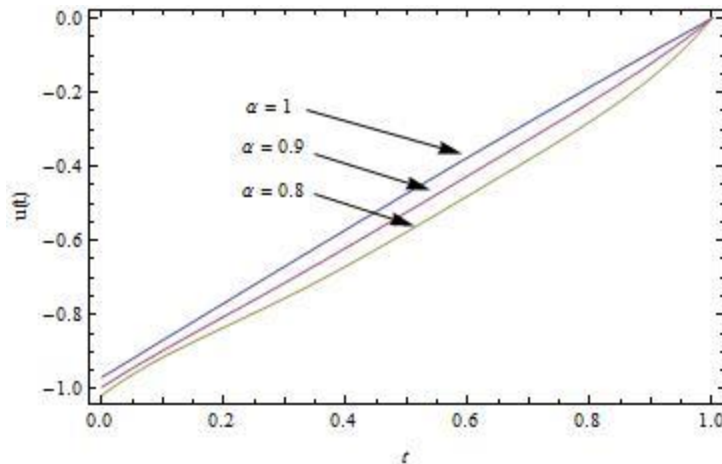


Fig 6. Approximate solutions of $u(t)$ for $m = 5$ and different values of α in example 2.

8. Conclusion

In this work, by Bernstein polynomials we obtained operational matrices of the product and fractional derivative. Then we reduced the time varying fractional optimal control problem to a system of algebraic equations that can be solved easily. We saw that the obtained results in examples were in good agreement with the exact solution and approximate solution of other methods. Also, we observed that the solutions for the fractional optimal control problems approach to the solutions for standard optimal control problems as the order of the fractional derivative approaches to 1.

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