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The Randers β-Change of More Generalized m-th Root Metrics

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Abstract

A change of Finsler metric $F(x,y) \to \bar{F}(x,y)$ is called a Randers β-change of F, if $\bar{F}(x,y) = F(x,y) + \beta(x,y)$, where $\beta(x,y) = b_i(x)y^i$ is a one-form on a smooth manifold M. The purpose of the present paper is devoted to studying the conditions for more generalized m-th root metrics $\tilde{F}_1 = \sqrt{{A_1}^{\frac{2}{m_1}} + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt{{A_2}^{\frac{2}{m_2}} + B_2 + C_2}$, when is established Randers β-change.

Keywords: m-th root metric; more generalized m-th root metric; Randers β-change

1. Introduction

Finsler geometry was first introduced locally by Finsler himself, to be studied by many eminent mathematicians for its theoretical importance and applications in the variational calculus, mechanics and theoretical physics.

Let (M, F) be an n-dimensional Finsler manifold. For a differential one-form β(x,y) = $b_i(x)y^i$ on M, G. Randers [1], in 1941, introduced a special Finsler space defined by the β-change

$$\bar{F} = F + \beta \tag{1.1}$$

where F is Riemannian. M. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which F is Finslerian.

In 1979, Shimada [3] introduced the m-th root metric on the differentiable manifold M defined as:

$$F = \sqrt[m]{a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}} \tag{1.2}$$

Where the coefficients $a_{i_1 i_2 \dots i_m}$ are the components of symmetric covariant tensor field of order (0,m) being the functions of positional co-ordinates only. Since then various geometers such as [4], [5] etc. have explored the theory of m-th root metric and studied its transformations. There exist the following important one class of Finsler metric,

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$$\tilde{F} = \sqrt{A^{\frac{2}{m}} + B} + \mathrm{C} \quad (1.3)$$

where A = $a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, B = $b_{ij}(x) y^i y^j$ and C = $c_k(x) y^k$. This form is called more generalized m-th root metric. Obviously, it is not reversible Finsler metric.

In this paper, we have considered a transformation of the more generalized m-th root metric such that it transforms to a similar metric as the Randers one defined in (1.1) in a way that the Riemannian metric F is replaced with more generalized m-th root metric e F defined in (1.3). Then, we obtain the conditions among two more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ due to Randers β-change. In overall this paper,

$$A_1 = a_{i_1 i_2 \dots i_{m_1}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_1}}, \quad (1.4)$$

$$A_2 = \bar{a}_{i_1 i_2 \dots i_{m_2}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_2}},$$

$$\mathrm{B}_1 = b_{ij}(x) y^i y^j, \ \mathrm{B}_2 = \bar{b}_{ij}(x) y^i y^j$$

$$\mathrm{C}_1 = c_k(x) y^k, \mathrm{C}_2 = \bar{c}_k(x) y^k,$$

and m_1, m_2 are belongs to natural numbers.

2. Main results

Case 1. m_1, m_2 are even numbers and $m_1 = m_2$.

Theorem 2.1: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are even numbers with $m_1 = m_2$ and $B_1 = B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $A_1 = \pm A_2$ and $C_1 = C_2 + \beta$.

Proof: For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} + C_2 + \beta. \quad (2.1)$$

By putting (-y) instead of (y) in (2.1), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} - C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} - C_2 - \beta. \quad (2.2)$$

Summing sides the two equations (2.1) and (2.2), we have

$$A_1^{\frac{2}{m}} + B_1 = A_2^{\frac{2}{m}} + B_2. \quad (2.3)$$

Consequently, we get the proof.■

We have the following.

Corollary 2.1: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are even numbers with $m_1 = m_2$ and $B_1 = B_2$. If $\sqrt[m_1]{A_1}$ and $\sqrt[m_2]{A_2}$ are Riemannian metrics, then $\tilde{F}_1 = \tilde{F}_2$ iff $C_1 = C_2$.

Case 2. m_1, m_2 are odd numbers and $m_1 = m_2$.

Theorem 2.2: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are odd numbers with $m_1 = m_2$ and $B_1 = B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $A_1 = \pm A_2$, $\pm i A_2$ and $C_1 = C_2 + \beta$.

Proof: For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} + C_2 + \beta. \qquad (2.4)$$

By putting (-y) instead of (y) in (2.4), we have

$$\sqrt{-A_1^{\frac{2}{m}} + B_1} - C_1 = \sqrt{-A_2^{\frac{2}{m}} + B_2} - C_2 - \beta. \qquad (2.5)$$

Summing sides the two equations (2.4) and (2.5), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + \sqrt{-A_1^{\frac{2}{m}} + B_1} = \sqrt{A_2^{\frac{2}{m}} + B_2} + \sqrt{-A_2^{\frac{2}{m}} + B_2}. \qquad (2.6)$$

Thus

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m}}}. \qquad (2.7)$$

Because of $B_1 = B_2$, $A_1 = \pm A_2$, $\pm i A_2$ and then $C_1 = C_2 + \beta$. ■

Theorem 2.3: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are odd numbers with $m_1 = m_2 (= m)$ and $B_1 \neq B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $m = 1$.

Proof: From (2.7), we have

$$2B_1B_2 = A_1^{\frac{4}{m}} + A_2^{\frac{4}{m}} + 2\sqrt{(B_1B_2)^2 - (B_2)^2 A_1^{\frac{4}{m}} - (B_1)^2 A_2^{\frac{4}{m}} + (A_1A_2)^{\frac{4}{m}}}. \qquad (2.8)$$

By (1.4), one can see that $m = 1$. ■

Case 3. m_1, m_2 are even numbers and $m_1 \neq m_2$.

Theorem 2.4: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are even numbers with $m_1 \neq m_2$, $m_1 > m_2$ and $B_1 = B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $A_1 = \pm \sqrt[m_2]{A_2^{m_1}}$ and $C_1 = C_2 + \beta$.

Proof: Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2 + \beta. \qquad (2.9)$$

By putting (-y) instead of (y) in (2.9), we have

$$\sqrt{-A_1^{\frac{2}{m_1}} + B_1} - C_1 = \sqrt{-A_2^{\frac{2}{m_2}} + B_2} - C_2 - \beta. \qquad (2.10)$$

Summing sides the two equations (2.9) and (2.10), we have

$$A_1^{\frac{2}{m_1}} + B_1 = A_2^{\frac{2}{m_2}} + B_2. \qquad (2.11)$$

Consequently, we get the proof. ■

In above theorem, if $m_1 - m_2 = k$, where k is even number, then by (2.11), we get

:(a_1) If $\frac{k}{m_2} > 1$, then

Case 1: $\frac{k}{m_2} = 2t$. Therefore, from theorem 2.4, $A_1 = \pm A_2^{1+2t}$.

Case 2: $\frac{k}{m_2} = 2t + 1$. Therefore, from theorem 2.4, $A_1 = \pm A_2^{2(1+t)}$.

Case 3: $m_2 \nmid k$. Because of $k = m_2 q + r$, from theorem 2.4, $A_1 = \pm A_2^{1+q+\frac{r}{m_2}}$.

:(a_2) If $\frac{k}{m_2} = 1$, then, from theorem 2.4, $A_1 = \pm (A_2)^2$.

:(a_3) If $\frac{k}{m_2} < 1$, then

Case 1: $\frac{m_2}{k} = 2t$. Therefore, from theorem 2.4, $A_1 = \pm A_2^{\frac{1+2t}{2t}}$.

Case 2: $\frac{m_2}{k} = 2t + 1$. Therefore, from theorem 2.4, $A_1 = \pm A_2^{\frac{2+2t}{1+2t}}$.

Case 3: $k \nmid m_2$. Because of $m_2 = k\acute{q} + \acute{r}$, from theorem 2.4, $A_1 = \pm A_2^{1+\frac{k}{k\acute{q}+\acute{r}}}$.

Case 4. m_1, m_2 are odd numbers and $m_1 \neq m_2$.

Theorem 2.5: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that

m_1, m_2 are odd numbers with $m_1 \neq m_2$, $m_1 > m_2$ and $B_1 = B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $A_1 = \pm A_2^{\frac{m_1}{m_2}}$, $A_1 = \pm i A_2^{\frac{m_1}{m_2}}$ and $C_1 = C_2 + \beta$.

Proof: Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2 + \beta. \qquad (2.12)$$

By putting (-y) instead of (y) in (2.12), we have

$$\sqrt{-A_1^{\frac{2}{m_1}} + B_1} - C_1 = \sqrt{-A_2^{\frac{2}{m_2}} + B_2} - C_2 - \beta. \qquad (2.13)$$

Summing sides the two equations (2.12) and (2.13), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + \sqrt{-A_1^{\frac{2}{m_1}} + B_1} = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + \sqrt{-A_2^{\frac{2}{m_2}} + B_2}\,. \qquad (2.14)$$

Thus

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m_1}}} = B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m_2}}}. \qquad (2.15)$$

Because of $B_1 = B_2$, we get $A_1 = \pm A_2^{\frac{m_1}{m_2}}$, $A_1 = \pm i A_2^{\frac{m_1}{m_2}}$ and then $C_1 = C_2 + \beta$. ■

Case 5. m_1, m_2 are even and odd numbers, respectively.

Theorem 2.6: Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$, where A_1, A_2, B_1, B_2, C_1 and C_2 are given by (1.4). Suppose that m_1, m_2 are even and odd numbers, respectively, and $B_1 = B_2$. If $\tilde{F}_1$ is Randers β-change of $\tilde{F}_2$, then $A_1 = \pm \sqrt[\frac{m_1}{2}]{\frac{1}{2}(-B_1 \pm \sqrt{(B_1)^2 - A_2^{\frac{4}{m_2}}})}$ and $C_1 = C_2 + \beta$.

Proof: Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2 + \beta. \qquad (2.16)$$

By putting (-y) instead of (y) in (2.16), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} - C_1 = \sqrt{-A_2^{\frac{2}{m_2}} + B_2} - C_2 - \beta. \qquad (2.17)$$

Summing sides the two equations (2.16) and (2.17), we have

$$2\sqrt{A_1^{\frac{2}{m_1}} + B_1} = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + \sqrt{-A_2^{\frac{2}{m_2}} + B_2}\,. \qquad (2.18)$$

Thus

$$4A_1^{\frac{4}{m_1}} + 4(B_1)^2 - 4B_1B_2 + (B_2)^2 + 4A_1^{\frac{2}{m_1}}(2B_1 - B_2) = (B_2)^2 - A_2^{\frac{4}{m_2}}. \quad (2.19)$$

Because of $B_1 = B_2$, we have

$$4A_1^{\frac{4}{m_1}} + 4B_1A_1^{\frac{2}{m_1}} + A_2^{\frac{4}{m_2}} = 0. \quad (2.20)$$

Consequently, $A_1 = \pm \sqrt[\frac{m_1}{2}]{\frac{1}{2}(-B_1 \pm \sqrt{(B_1)^2 - A_2^{\frac{4}{m_2}}})}$ and then $C_1 = C_2 + \beta$. ■

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