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Spectrum Preserving Linear map on Liminal C*-Algebras

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Abstract

Let A and B be unital semi-simple Banach algebras. If B is a liminal C*-algebra and φ is a surjective spectrum preserving linear mapping from A to B, then φ is a Jordan homomorphism.

Keywords: Semi-simple, liminal C*-algebra, Spectrum preserving, Jordan homomorphism.

1. Introduction

In 1970, Kaplansky [8] asked the following question:
Let $\varphi: A \to B$ be a unital, invertibility preserving linear map between unital Banach algebras A and B. Is φ a Jordan homomorphism?
Let us quickly state that the above question of Kaplansky is too general and the answer to it is negative in this generality. Sourour [2] proved that if φ is not surjective, then it may not be a Jordan homomorphism. Aupetit [9] showed that Kaplansky's question may not have a positive answer if the Banach algebras A and B are not semi-simple. In view of the above discussions, the following conjecture seems quite natural.
Conjecture: Suppose A and B are unital semisimple Banach algebras and $\varphi: A \to B$ is a unital surjective linear map preserving invertibility. Is φ a Jordan homomorphism?
In this generality, the problem remains unsolved. A number of partial positive results have been found, especially in the case where the map is spectrum preserving. Jafarian and Sourour proved that in [1] every surjective, spectrum preserving linear map from B(X) onto B(Y) is an isomorphism or an anti-isomorphism. Aupetit [3] proved that every surjective, spectrum preserving linear map from a Von-

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Neumann algebra onto another Von Neumann algebra is a Jordan isomorphism. For more information one can see [4].
In this paper we will prove that every surjective spectrum preserving linear map between semi-simple unital Banach algebras A and B is a Jordan homomorphism provided that one of A or B is liminal.

2. Preliminaries

Let A be a complex unital normed algebra. The spectrum of an element a of A will be denoted by σ(a).
Let A and B are Banach algebras. A linear map $\varphi: A \to B$ is said to be spectrum preserving if for all $x \in A$ $\sigma(\varphi(x)) = \sigma(x)$ and a Jordan homomorphism if $\varphi(x^2) = \varphi(x)^2$ for all $x \in A$.
A C*-algebra A is said to be liminal if for every non-zero irreducible representation (H, φ) of A we have $\varphi(A) = K(H)$.

2.1. Lemma

Let A and B be Banach algebras with B semisimple. If $\varphi: A \to B$ is a surjectivet, spectrum preserving linear map, then $\varphi(1) = 1$ [3].

2.2. Lemma

Let A and B be two semi-simple Banach algebras. If φ is a spectrum preserving linear mapping from A into B. Then φ is injective[3].

2.3. Lemma

Let A and B be two semi-simple Banach algebras. If $\varphi: A \to B$ is a surjective, spectrum preserving linear map, then φ^{-1} is spectrum preserving.

Proof

Since φ is a surjective, spectrum preserving linear map by Lemma 2.2, φ is injective, so φ is invertible. If $b \in B$, then there exists $a \in A$ such that $\varphi(a) = b$. Therefor $\sigma\left(\varphi^{-1}(b)\right) = \sigma(a) = \sigma\left(\varphi(a)\right) = \sigma(b)$.

3. Main Result

In the fallowing we show that every surjective spectrum preserving linear map between semi-simple unital Banach algebras A and B is a Jordan homomorphism if A or B is liminal.

3.1. Theorem

Every unital liminal C*-algebra A has only finite dimensional irreducible representations.

Proof

If (H, φ) is a non-zero irreducible representation of A, then it is non-degenerate, so $\varphi(1) = id_H$. Hence id_H is compact, and therefore $\dim(H) < \infty$.

3.2. Corollary

Every unital liminal C*-algebra A has a separating family of finite dimensional irreducible representations.

Proof

Since every unital liminal C*-algebra is semi-simple, and the radical of A is the intersection the kernels of all the irreducible representations of A, so for every nonzero element a of A, there exists an irreducible representation π of A such that $\pi(a) \neq 0$. By Theorem 3.1, π has finite dimensional.

3.3. Theorem

Let A be semi-simple Banach algebra and B a liminal C*-algebra both with identity. If φ is a surjective spectrum preserving linear mapping from A to B, then φ is a Jordan homomorphism.

Proof

Since B is unital luminal C*-algebra by Corollary 3.2, B has a family of finite dimensional irreducible representations. Since φ is a linear surjective and invertibility preserving map then there exists a Jordan homomorphism S such that $\varphi(x) = \varphi(1)S(x)$ for all element x of A [5, Theorem 2]. On the other hand, Lemma 2.1 implies that $\varphi(1) = 1$. So we have $\varphi(x) = S(x)$ for all $x \in A$, therefor φ is a Jordan homomorphism.

3.4. Corollary

Let A and B be unital semi-simple Banach algebras and A is liminal C*-algebra. If φ is a surjective spectrum preserving linear mapping from A to B, then φ is a Jordan homomorphism.

Proof

Since φ is a surjective, spectrum preserving linear map by Lemma 2.2, φ is injective, so φ is invertible and by Lemma 2.3, the map φ^{-1} preservs invertibility. Then by Theorem 3.3, φ^{-1} is a Jordan homomorphism and therefore, φ is Jordan homomorphism.
In the following we give an example of C*-algebras which is liminal but is not Vonneumann, so our result may cover some other classes of algebras that the above conjecture could be true.

3.5. Example

Let $T = [0, \infty]$. Define the C*- subalgebra X of M_2 as follows:

$$X = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \middle| \lambda \in C \right\}$$

Let $A = \{f \in C(T, M_2) | f(\infty) \in X\}$. Since A has a unit element and not abelian so A is not a Von Neumann algebra [7, Theorem 4.1.5]. Also every irreducible representation of $C(T, M_2)$ is of dimension 2 [6,pp.330] so $C(T, M_2)$ is liminal [6, Example IV.1.3.3]. Therefore, A is liminal [7, Theorem 5.6.1].

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