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Coupled fixed point theorems in partially ordered ε-chainable Metric spaces

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Abstract

In this paper, we introduce the notion of partially ordered ε-chainable metric spaces and we derive new coupled fixed point theorems for uniformly locally contractive mappings on such spaces.

Keywords: Coupled fixed point, ε-chainable, uniformly locally contractive, partially ordered set, mixed monotone property

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1. Introduction

The Banach fixed point theorem [4] is a simple and powerful theorem with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, and integral equations. This theorem has been generalized and extended by many authors in various ways; see ([1-3], [5]-[24]) and others.

Recently, Ran and Reurings [20], Bhaskar and Lakshmikantham [9], Nieto and Lopez [18], Agarwal, El-Gebeily and O'Regan [1] and Lakshmikantham and Ciric [11] presented some new results for contractions in partially ordered metric spaces (see also [3], [5], [6], [10], [12-17], [19], [21]). For a given partially ordered set X, Bhaskar and Lakshmikantham in [9] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$. Later in [11] Lakshmikantham and Ciric investigated some more coupled fixed point theorems in partially ordered sets. Very recently, Samet [21] extended the results of Bhaskar and Lakshmikantham [9] to mappings satisfying a generalized Meir-Keeler contractive condition.

In this paper, we introduce the notion of partially ordered ε-chainable metric spaces and we derive new coupled fixed point theorems for uniformly locally contractive mappings on such spaces. To begin, we first recall some definitions given in [9] which will be used in this work.

Definition 1.1. Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. We say that F has the mixed monotone property if for any $x, y \in X$, we have:

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$
$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$$

Definition 1.2. Let X be a non-empty set and $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if: $F(x, y) = x$ and $F(y, x) = y$.

Now, we introduce the following definitions.

Definition 1.3. Let $(X, \leq)$ be a partially ordered set endowed with a metric d and $\varepsilon > 0$. We say that X is ε-chainable with respect to the partial order $\leq$ on X, if for any two points $a, b \in X$ such that $a \leq b$, there exists a finite set of points:

$$a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n = b$$

such that $d(\alpha_{i-1}, \alpha_i) < \varepsilon$ for all $i = 1, 2, \cdots, n$.

Definition 1.4. Let $(X, \leq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping. We say that F is (ε, λ) uniformly locally contractive if:

$$\frac{d(x,u) + d(y,v)}{2} < \varepsilon \Rightarrow d(F(x, y), F(u,v)) < \frac{\lambda}{2}[d(x,u) + d(y,v)], \forall x \geq u, \forall y \leq v,$$

where $\varepsilon > 0$ and $\lambda \in (0,1)$.

Through this paper, we will use the following notations. Let $(X, \leq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping.

- We endow the product space $X \times X$ with the partial order $\leq$ defined by:

$$(x, y), (u,v) \in X \times X, (u,v) \leq (x, y) \Leftrightarrow x \geq u, y \leq v.$$

- We endow the product space $X \times X$ with the metric η defined by:

$$\eta((x,y),(u,v)) = d(x,u) + d(y,v), \forall (x,y),(u,v) \in X \times X .$$

- For all $(x,y) \in X \times X$, we denote:

$$F^0(x,y) = x,\ F^1(x,y) = F(x,y),\ F^{m+1}(x,y) = F(F^m(x,y), F^m(y,x))\ \forall m \in N .$$

Here, N is the set of all positive integers.

2. Main results

The following lemma is the principal tool used to prove the main results.

Lemma 2.1. Let $(X,\leq)$ be a partially ordered set endowed with a metric d and $F: X \times X \to X$ be a given mapping. We assume that

1. X is ε-chainable with respect to the partial order $\leq$ on X,
2. F has the mixed monotone property,
3. F is (ε,λ) uniformly locally contractive mapping,
4. $\exists\ (a,b),(a^*,b^*) \in X \times X$ such that $a \leq b$ and $a^* \geq b^*$.

Then,

$$\lim_{m\to+\infty} \eta((F^m(a,a^*), F^m(a^*a)),(F^m(b;b^*), F^m(b^*,b))) = 0 . \qquad (1)$$

Moreover, we have:

$$\eta((F^m(a,a^*), F^m(a^*a)),(F^m(b;b^*), F^m(b^*,b))) < 2n\lambda^m,\ \forall m .$$

Proof. Since X is ε-chainable, there exist $\alpha_0, \alpha_1, \cdots, \alpha_n \in X$ and $\beta_0, \beta_1, \cdots, \beta_n \in X$ such that

$$\begin{cases} a = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n = b \\ d(\alpha_{i-1}, \alpha_i) < \varepsilon, \forall i = 1,2,\cdots,n \end{cases} \qquad (2)$$

and

$$\begin{cases} b^* = \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_1 \leq \beta_0 = a^* \\ d(\beta_{i-1}, \beta_i) < \varepsilon, \forall i = 1,2,\cdots,n. \end{cases} \qquad (3)$$

From (2)-(3) and using the mixed monotone property of F, we can show easily that for all i, we have:

$$F^m(\alpha_i,\beta_i) \geq F^m(\alpha_{i-1},\beta_{i-1}) \text{ and } F^m(\beta_i,\alpha_i) \leq F^m(\beta_{i-1},\alpha_{i-1}) \text{ for all } m \in N . \qquad (4)$$

Now, we claim that for all $m \in N$, we have:

$$d(F^m(\alpha_i,\beta_i), F^m(\alpha_{i-1},\beta_{i-1})) < \lambda^m \varepsilon \text{ and } d(F^m(\beta_i,\alpha_i), F^m(\beta_{i-1},\alpha_{i-1})) < \lambda^m \varepsilon . \qquad (5)$$

To prove (5), we will argue by induction. This result is trivial for $m=0$. Let us check that (5) is true for $m=1$. From (2)-(3), we get:

$$\frac{d(\alpha_i,\alpha_{i-1})+d(\beta_i,\beta_{i-1})}{2}<\varepsilon, \quad \frac{d(\beta_{i-1},\beta_i)+d(\alpha_{i-1},\alpha_i)}{2}<\varepsilon.$$

Since $\alpha_i \geq \alpha_{i-1}$, $\beta_i \leq \beta_{i-1}$ and F is (ε,λ) uniformly locally contractive, we obtain:

$$d(F\ (\alpha_i,\beta_i),F\ (\alpha_{i-1},\beta_{i-1}))<\lambda\ \varepsilon \text{ and } d(F\ (\beta_i,\alpha_i),F\ (\beta_{i-1},\alpha_{i-1}))<\lambda\ \varepsilon.$$

Then, (5) is true for $m=1$. Now, assume that (5) holds for a given $m\in N$. Let us prove that

(5) holds also for $m+1$.

Since (5) holds for m, we get:

$$\frac{d(F^m(\alpha_i,\beta_i),F^m(\alpha_{i-1},\beta_{i-1}))+d(F^m(\beta_i,\alpha_i),F^m(\beta_{i-1},\alpha_{i-1}))}{2}<\lambda^m\varepsilon$$

$$\frac{d(F^m(\beta_{i-1},\alpha_{i-1}),F^m(\beta_i,\alpha_i))+d(F^m(\alpha_{i-1},\beta_{i-1}),F^m(\alpha_i,\beta_i))}{2}<\lambda^m\varepsilon.$$

Then, from (4) and since F is (ε,λ) uniformly locally contractive, we obtain:

$$\begin{cases} d(F(F^m(\alpha_i,\beta_i),F^m(\beta_i,\alpha_i)),F(F^m(\alpha_{i-1},\beta_{i-1}),F^m(\beta_{i-1},\alpha_{i-1})))<\lambda^{m+1}\varepsilon \\ d(F(F^m(\beta_{i-1},\alpha_{i-1}),F^m(\alpha_{i-1},\beta_{i-1})),F(F^m(\beta_i,\alpha_i),F^m(\alpha_i,\beta_i)))<\lambda^{m+1}\varepsilon \end{cases}$$

which implies that (5) holds for $m+1$. Then, (5) holds for all $m\in N$. Now, using the triangular inequality and (5), we get:

$$d(F^m(a,a^*),F^m(b,b^*))\leq d(F^m(\alpha_0,\beta_0),F^m(\alpha_1,\beta_1))+\cdots+d(F^m(\alpha_{n-1},\beta_{n-1}),F^m(\alpha_n,\beta_n))<n\lambda^m\varepsilon \qquad (6)$$

Similarly, one can show that

$$d(F^m(a^*,a\ ),F^m(b^*,b))<n\lambda^m\varepsilon. \qquad (7)$$

Combining (6) and (7) and using that $\lambda\in(0,1)$, we obtain:

$$\eta((F^m(a,a^*),F^m(a^*a)),(F^m(b;b^*),F^m(b^*,b)))<2n\lambda^m\to 0 \text{ as } m\to+\infty.$$

This makes end to the proof.

Now, we are able to prove some theorems. We start by studying the existence of a coupled fixed point. Our first result is the following.

Theorem 2.1. Let $(X,\leq)$ be a partially ordered set endowed with a metric d such that (X,d) is complete. Let $F: X\times X\to X$ be a given mapping. We assume that

1. X is ε -chainable with respect to the partial order $\leq$ on X ,
2. F is continuous,
3. F has the mixed monotone property,
4. F is (ε,λ) uniformly locally contractive mapping,
5. $\exists\ x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

Then, F admits a coupled fixed point.

Proof. Let us define the sequences $\{x_m\}$ and $\{y_m\}$ in X by :

$$\begin{cases} x_{m+1} = F^{m+1}(x_0, y_0) = F(F^m(x_0, y_0), F^m(y_0, x_0)), \\ y_{m+1} = F^{m+1}(y_0, x_0) = F(F^m(y_0, x_0), F^m(x_0, y_0)). \end{cases}$$

By taking $(a,b) = (x_0, F(x_0, y_0)) = (x_0, x_1)$ and $(a^*, b^*) = (y_0, F(y_0, x_0)) = (y_0, y_1)$, we show that all the hypotheses required by Lemma 2.1 are satisfied. Hence,

$$A_m := \eta((F^m(x_0, y_0), F^m(y_0, x_0)), (F^m(x_1, y_1), F^m(y_1, x_1))) < 2n\lambda^m\varepsilon. \qquad (8)$$

Now, we will show that $\{x_m\}$ and $\{y_m\}$ are Cauchy sequences in X . We have :

$$d(x_m, x_{m+1}) = d(F^m(x_0, y_0), F^{m+1}(x_0, y_0)) = d(F^m(x_0, y_0), F^m(x_1, y_1)) \leq A_m .$$

Then, from (8), it follows immediately that $\{x_m\}$ is a Cauchy sequence in X . Similarly, we have :

$$d(y_m, y_{m+1}) = d(F^m(y_0, x_0), F^{m+1}(y_0, x_0)) = d(F^m(y_0, x_0), F^m(y_1, x_1)) \leq A_m$$

and $\{y_m\}$ is also a Cauchy sequence in X .

Now, since (X,d) is a complete metric space, there exists $(x,y) \in X \times X$ such that

$$x_m \xrightarrow{d} x \quad \text{and} \quad y_m \xrightarrow{d} y \ \text{ as } m \to +\infty . \qquad (9)$$

With the continuity of F , (9) implies that

$$F(x_m, y_m) \xrightarrow{d} F(x,y) \ \text{ and } \ F(y_m, x_m) \xrightarrow{d} F(y,x) \ \text{ as } m \to +\infty . \qquad (10)$$

Using the triangular inequality, (9) and (10), we obtain:

$$d(x, F(x,y)) \leq d(x, x_{m+1}) + d(F(x_m, y_m), F(x,y)) \to 0 \ \text{ as } m \to +\infty .$$

This emplies that $F(x, y) = x$. Similarly, we have :

$$d(y, F(y, x)) \le d(y, y_{m+1}) + d(F(y_m, x_m), F(y, x)) \to 0 \text{ as } m \to +\infty .$$

Then, $F(y, x) = y$. Finally, (x, y) is a coupled fixed point of F . This makes end to the proof.

As it is showed in [9], if we require that the underlying metric space X has an additional property, the previous result is still valid for F not necessarily continuous. We discuss this in the following theorem.

Theorem 2.2. Let $(X, \le)$ be a partially ordered set endowed with a metric d such that (X, d) is complete. Let $F : X \times X \to X$ be a given mapping. We assume that

1. X is ε-chainable with respect to the partial order $\le$ on X ,
2. if $\{x_m\}$ is a nondecreasing sequence in X such that $x_m \xrightarrow{d} x$ as $m \to +\infty$, then $x_m \le x$ for all m ,
3. if $\{y_m\}$ is a nonincreasing sequence in X such that $y_m \xrightarrow{d} y$ as $m \to +\infty$, then $y_m \ge y$ for all m ,
4. F has the mixed monotone property,
5. F is (ε, λ) uniformly locally contractive mapping,
6. $\exists\ x_0, y_0 \in X$ such that $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$.

Then, F admits a coupled fixed point.

Proof. Following the proof of Theorem 2.1, we have only to show that (x, y) is a coupled fixed point of F . Let $p > 1$. From 1 and 2, there exists $m(p) \in N$ such that

$$\begin{cases} \dfrac{d(x, x_{m(p)}) + d(y, y_{m(p)})}{2} = \dfrac{d(y_{m(p)}, y) + d(x_{m(p)}, x)}{2} < \dfrac{\varepsilon}{p} (< \varepsilon), \\ d(x, x_{m(p)+1}) < \dfrac{\varepsilon}{p}, d(y, y_{m(p)+1}) < \dfrac{\varepsilon}{p}. \end{cases} \quad (11)$$

From the hypothesis 4, it is clear that $\{x_m\}$ is a nondecreasing sequence and $\{y_m\}$ is a nonincreasing sequence. Then, from hypotheses 2 and 3, we have:

$$x_m \le x \text{ and } y_m \ge y \text{ for all } m . \quad (12)$$

Since F is (ε,λ) uniformly locally contractive, from (11)-(12), we get:

$$d(F(x,y),F(x_{m(p)},y_{m(p)}))<\frac{\lambda\varepsilon}{p} \text{ and } d(F(y_{m(p)},x_{m(p)}),F(y,x))<\frac{\lambda\varepsilon}{p}. \qquad (13)$$

Using the triangular inequality, from (11) and (13), we obtain :

$$d(x,F(x,y))\le d(x,x_{m(p)+1})+d(F(x_{m(p)},y_{m(p)}),F(x,y))<\frac{\varepsilon(\lambda+1)}{p}\to 0 \text{ as } p\to+\infty .$$

Hence, $x=F(x,y)$. By a similar argument, we can show that $y=F(y,x)$. Finally, (x,y) is a coupled fixed point of F and the proof is completed.

One can prove that the coupled fixed point is in fact unique, provided that the product space $X\times X$ endowed with the partial order mentioned earlier has the following property:

(H): $\forall(x,y),(x^*,y^*)\in X\times X, \exists(z_1,z_2)\in X\times X$ that is comparable to (x,y) and (x^*,y^*).

This is the purpose of the next theorem.

Theorem 2.3. Adding condition (H) to the hypotheses of Theorem 2.1, we obtain the uniqueness of the coupled fixed point of F .

Proof. Assume that (x^*,y^*) is another coupled fixed point of F . We distinguish two cases.

First case: (x,y) and (x^*,y^*) are comparable with respect to the ordering in $X\times X$. Without restriction to the generality we can assume that $x\le x^*$ and $y\ge y^*$. Applying Lemma 2.1, we get:

$$\lim_{m\to+\infty}\eta((F^m(x,y\ \ ),F^m(y,x)),(F^m(x^*;y^*),F^m(y^*,x^*)))=0$$

On the other hand, for all $m\in N$, we have:

$$x=F^m(x,y),\ y=F^m(y,x),\ x^*=F^m(x^*,y^*),\ y^*=F^m(y^*,x^*).$$

Then, $\eta((x,y),(x^*,y^*))=0$ and $(x,y)=(x^*,y^*)$.

Second case: (x,y) and (x^*,y^*) are not comparable. From (H), there exists $(z_1,z_2)\in X\times X$ that is comparable to (x,y) and (x^*,y^*). Without restriction to the generality, we can suppose that $x\le z_1,\ y\ge z_2$ and $x^*\le z_1,\ y^*\ge z_2$. Again, applying Lemma 2.1, we get:

$$\begin{cases}\lim_{m\to+\infty}\eta((F^m(x,y),F^m(y,x)),(F^m(z_1,z_2),F^m(z_2,z_1)))=0,\\ \lim_{m\to+\infty}\eta((F^m(x^*,y^*),F^m(y^*,x^*)),(F^m(z_1,z_2),F^m(z_2,z_1)))=0.\end{cases} \qquad (14)$$

Now, using the triangular inequality and (14), we obtain:

$$\eta((x,y),(x^*,y^*))=\eta((F^m(x,y),F^m(y,x)),(F^m(x^*,y^*),F^m(y^*,x^*)))$$

$$\leq \eta((F^m(x,y),F^m(y,x)),(F^m(z_1,z_2),F^m(z_2,z_1)))$$

$$+\eta((F^m(z_1,z_2),F^m(z_2,z_1)),(F^m(x^*,y^*),F^m(y^*,x^*)))$$

$$\to 0 \text{ as } m \to +\infty.$$

Then, $\eta((x,y),(x^*,y^*))=0$ and $(x,y)=(x^*,y^*)$. This makes end to the proof.

Now, we will prove the following result.

Theorem 2.4. In addition to the hypotheses of Theorem 2.1, suppose that every pair of elements of X has an upper or a lower bound in X. Then, $x = y$.

Proof. We distinguish two cases.

<u>First case:</u> $x = F^m(x,y)$ is comparable to $y = F^m(y,x)$. We can assume that $x \leq y$. We can write:

$$x \leq y \text{ and } y \geq y.$$

Applying Lemma 2.1, we get:

$$\lim_{m\to+\infty} B_m := \eta((F^m(x,y),F^m(y,x)),(F^m(y,y),F^m(y,y)))=0. \tag{15}$$

From (15), we get:

$$d(x,y)=d(F^m(x,y),F^m(y,x)) \leq d(F^m(x,y),F^m(y,y))+d(F^m(y,y),F^m(y,x))$$
$$= B_m \to 0 \text{ as } m \to +\infty.$$

Then, $x = y$.

<u>Second case :</u> x is not comparable to y. Then, there exists an upper bound or lower bound of x and y. That is, there exists $z \in X$ comparable with x and y. For example, we can suppose that $x \leq z$ and $y \leq z$. Again, applying Lemma 2.1, we obtain:

$$\lim_{m\to+\infty} C_m := \eta((F^m(x,y),F^m(y,x)),(F^m(z,z),F^m(z,z)))=0. \tag{16}$$

From (16), we get:

$$d(x,y)=d(F^m(x,y),F^m(y,x)) \leq d(F^m(x,y),F^m(z,z))+d(F^m(z,z),F^m(y,x))$$
$$= C_m \to 0 \text{ as } m \to +\infty.$$

Then, $x = y$ and the proof is completed.

Alternatively, if we know that the elements x_0 and y_0 are such that $x_0 \leq y_0$, then we can also demonstrate that the components x and y of the coupled fixed point are indeed the same. This is the purpose of the next theorem.

Theorem 2.5. In addition to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), suppose that $x_{0,} y_0 \in X$ are comparable. Then, $x = y$.

Proof. Without restriction to the generality, we can assume that $x_0 \leq y_0$. Applying Lemma 2.1, we get:

$$\lim_{m\to+\infty} D_m := \eta((F^m(x_0,y_0),F^m(y_0,x_0)),(F^m(y_0,x_0),F^m(x_0,y_0))) = 0. \qquad (17)$$

From (17) and using the triangular inequality, we get:

$$\begin{aligned} d(x,y) &\leq d(x,x_m) + d(x_m,y_m) + d(y_m,y) \\ &= d(x_m,x) + d(F^m(x_0,y_0),F^m(y_0,x_0)) + d(y_m,y) \\ &\leq d(x_m,x) + D_m + d(y_m,y) \\ &\to 0 \text{ as } m \to +\infty. \end{aligned}$$

Then, $d(x,y) = 0$ and $x = y$. This makes end to the proof.

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