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Asymptotics of the eigenvalues for the operators with transition points and Neumann conditions

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Abstract

In this paper we consider the following differential equation

$$LW \equiv \frac{d^2W}{d\xi^2} - \left(\Psi(\xi) - u^2(1-\xi^2)\right) W = 0, \qquad \xi \in [a,b] \tag{1}$$

where $\xi\epsilon[a,b]$, $a = -b > 1$, u is a large parameter and $\Psi(\xi)$ is a continuous function on $[a,b]$. For equation (1), $\xi = \pm 1$ are transition points and $r(\xi) = 1 - \xi^2$ be a weight function. Using the asymptotic solution constructed by Olver in [3] and [5], we study the asymptotic behavior of the eigenvalues of the operator L with Neumann boundary conditions $W'(a) = W'(b) = 0$.

1 Introduction

In the paper we consider the differential equation (1) with Neumann boundary conditions. We will study the positive distribution of eigenvalues of operator L. Moreover we will show that the positive eigenvalues of equation (1) in two transition points case with Neumann boundary conditions depend only on the positive part of weight function. Differential equations with transition points play an important role in various areas of mathematics. For example, transition points connected with physical situations in which zeros correspond to the limit of motion of a wave, mechanical particle bound by a potential field. Transition points appear also in elasticity, optics, geophysics and other branch of natural sciences [see 6]. Moreover, a wide class of differential equations with Bessel-type singularities and their perturbations can be reduced to differential equations having turning points. we note that some aspects of the turning points theory and a number of its applications are described in [7]. It is known [3], that two linear independent solutions $U_1(u,\xi)$ and $U_2(u,\xi)$ of the equation

$$\frac{d^2U}{d\xi^2} = u^2(\xi^2-1)U \tag{2}$$

have for large u and $\xi > 0$ the following asymptotic forms.

$$U_1(u,\xi) = \sqrt[4]{4\pi}\left\{\Gamma(1\ \ 2+u/2)\right\}^{1/2} u^{-1/12}\left(\eta_\xi\ \ (\xi^2-1)\right)^{1/4}\left\{A_i(u^{2/3}\eta_\xi)+O(u^{-1})\right\} \tag{3}$$

$$U_2(u,\xi) = \sqrt[4]{4\pi}\left\{\Gamma(1\ \ 2+u/2)\right\}^{1/2} u^{-1/12}\left(\eta_\xi\ \ (\xi^2-1)\right)^{1/4}\left\{B_i(u^{2/3}\eta_\xi)+O(u^{-1})\right\}, \tag{4}$$

where,$\eta_\xi = \left\{3\ \ 2\int_1^\xi(\tau^2-1)^{1/2}d\tau\right\}^{2/3}$ $\xi \geq 1$ and $\eta_\xi = -\left\{3\ \ 2\int_1^\xi(1-tau^2)^{1/2}d\tau\right\}^{2/3}$ $0 \leq \xi \leq 1$. The functions $A_i(u^{2/3}\eta_\xi)$ and $B_i(u^{2/3}\eta_\xi)$ are two independent solutions of the equation $\frac{d^2\omega}{d\xi^2} = u^2\xi\omega(\xi)$. The connection formulas are

$$U_1(u,-\xi) = cos(\pi u/2)U_2(u,\xi) + sin(\pi u/2)U_1(u,\xi), \tag{5}$$

$$U_2(u,-\xi) = cos(\pi u/2)U_1(u,\xi) + sin(\pi u/2)U_2(u,\xi). \tag{6}$$

The Wronskian of two solutions $U_1(u,\xi)$,$U_2(u,\xi)$ is equal to

$$W\{U_1(u,\xi),U_2(u,\xi)\} = 2^{1/2}\pi^{-1/2}\Gamma(1\ \ 2+\pi u/2).$$

2 Expansion of solutions in descending power of the large parameter u

In this section we get the derivative in ξ of the solutions $W_1(u,\xi)$, $W_2(u,\xi)$ of the equation (1). In [3] the asymptotic solutions of the equation(1) in the form

$$W(u,\xi) = V(u,\xi)\sum_{s=0}^{\infty}\frac{A_s}{u^{2s}} + \frac{1}{u^2}\frac{\partial V(u,\xi)}{\partial\xi}\sum_{s=0}^{\infty}\frac{B_s(\xi)}{u^{2s}} \tag{7}$$

was established. Here $V(u,\xi) = U_1(u,\xi)$ or $U_2(u,\xi)$, $A_0 = 1$, $A_s(\xi)$ and $B_s(\xi)$ are defined in [3]. In the asymptotic solution, the derivative of $V(u,\xi)$ can be written in the following form

$$U_1'(a,\xi) = \frac{\xi}{2}U_1(a,\xi) - U_1(a-1,\xi), \qquad U_2'(a,\xi) = -\frac{\xi}{2}U_2(a,\xi) - U_2(a+1,\xi). \tag{8}$$

So, by substituting (8)in (7) for $\xi > 0$ we have

$$W_1(u,\xi) = U_1(u,\xi)K_1(\infty,\xi), \tag{9}$$

$$W_2(u,\xi)=U_2(u,\xi)K_2(\infty,\xi), \tag{10}$$

where, $K_1(n,\xi)$ and $K_2(n,\xi)$ are defined in [4]

$$K_1(n,\xi)=\sum_{s=0}^{n}A_s(\xi)u^{-2s}+\frac{1}{u^2}(u\xi-\sqrt{2u})\sum_{s=0}^{n-1}B_s(\xi)u^{-2s}, n=0,1,2,3,..., \tag{11}$$

$$K_2(n,\xi)=\sum_{s=0}^{n}A_s(\xi)u^{-2s}+\frac{1}{u^2}(-u\xi+\sqrt{2u})\sum_{s=0}^{n-1}B_s(\xi)u^{-2s}, n=0,1,2,3,.... \tag{12}$$

For $\xi<0$ using (5) and (6) the solutions $W_1(u,\xi)$ and $W_2(u,\xi)$ will be

$$W_1(u,\xi)=\left\{U_1(u,-\xi)sin\frac{\pi u}{2}+U_2(u,-\xi)cos\frac{\pi u}{2}\right\}K_1(\infty,-\xi), \tag{13}$$

$$W_2(u,\xi)=\left\{U_1(u,-\xi)cos\frac{\pi u}{2}-U_2(u,-\xi)sin\frac{\pi u}{2}\right\}K_2(\infty,-\xi). \tag{14}$$

In [3] the asymptotic behavior of W_1 and W_2 in the following forms

$$W_1(u,\xi)=U_1(\frac{-u}{2},\xi\sqrt{2u})(1+O(u^{-1})), \qquad W_2(u,\xi)=U_2(\frac{-u}{2},\xi\sqrt{2u})(1+O(u^{-1})) \tag{15}$$

was established. Now we want to get the derivation of asymptotic form of solutions. If $x=\xi\sqrt{2u}$ then by (8) we get

$$\frac{\partial U}{\partial \xi}=\sqrt{2u}\left(\frac{x}{2}U(a,x)-U(a-1,x)\right)=\sqrt{2u}\left\{\frac{\xi}{2}\sqrt{2u}U(a,\xi\sqrt{2u})-U(a-1,\xi\sqrt{2u})\right\}. \tag{16}$$

Now by using the solutions $U_1(u,\xi)$, $U_2(u,\xi)$ and substituting them in (16), we obtain

$$\begin{aligned}\frac{\partial U}{\partial \xi}&=\sqrt[4]{4\pi}\xi u\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\eta_\xi)\\&-\sqrt[4]{4\pi}\xi u\{\Gamma(\frac{1}{2}+\frac{u+2}{2})\}^{\frac{1}{2}}(u+2)^{\frac{-1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}}A_i\left((u+2)^{\frac{2}{3}}\eta_\xi\right).\end{aligned} \tag{17}$$

If we substitute $\frac{\partial U_1}{\partial \xi}$ from (17) in $\frac{\partial W_1(u,\xi)}{\partial \xi}$, then we get

$$\frac{\partial W_1(u,\xi)}{\partial \xi}=2\pi^{\frac{1}{4}}\sqrt{u}\left\{\Gamma(\frac{1}{2}+\frac{u}{2})\right\}^{\frac{1}{4}}u^{\frac{-1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}}(\frac{\xi\sqrt{2u}}{2}A_i(u^{\frac{2}{3}}\eta_\xi)-A_i\left((u+2)^{\frac{2}{3}}\eta_\xi\right)(1+O(u^{-1})). \tag{18}$$

For $\xi>0$ the asymptotic behavior of Airy functions A_i when $n\to\infty$ is given in [5]

$$A_i(u^{\frac{2}{3}}\eta_\xi)\sim\frac{e^{\frac{-2u\eta_\xi^{\frac{2}{3}}}{3}}}{2\sqrt[12]{\pi^6u^2\eta_\xi^3}}\sum_{s=0}^{n}(-1)^s u_s(\frac{2}{3}u\eta_\xi^{\frac{2}{3}})^{-s}, \qquad \eta>0$$

$$A_i((u+2)^{\frac{2}{3}}\eta_\xi) \sim \frac{e^{\frac{-2(u+2)\eta_\xi^{\frac{2}{3}}}{3}}}{2\sqrt[12]{\pi^6(u+2)^2\eta_\xi^3}}\sum_{s=0}^{n}(-1)^s u_s(\frac{2}{3}(u+2)\eta_\xi^{\frac{2}{3}})^{-s} \tag{19}$$
$$= \lambda(\xi)A_i(u^{\frac{2}{3}}\eta_\xi)(1+O(u^{-1})), \qquad \eta > 0.$$

Therefore if we put (19) in (18) then for $\frac{\partial W_1(u,\xi)}{\partial\xi}$ we have

$$\frac{\partial W_1(u,\xi)}{\partial\xi} = 2\sqrt{u}\pi^{\frac{1}{4}}\left\{\Gamma(\frac{1}{2}+\frac{u}{2})\right\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\eta_\xi)\left\{\frac{\xi\sqrt{2u}}{2}-\lambda(\xi)\right\}\left(1+O(u^{-1})\right).$$

In fact we have obtained the following results,

$$\frac{\partial W_1}{\partial\xi} \simeq W_1\left\{\xi u - \sqrt{2u}\mu(\xi)\right\}, \quad where \quad \mu(\xi) = e^{\frac{4}{3}}\eta_\xi^{\frac{2}{3}}, \tag{20}$$

and similarly $\frac{\partial W_2}{\partial\xi}$ can be written in the form

$$\frac{\partial W_2}{\partial\xi} \simeq W_2\left\{\xi u - \sqrt{2u}\lambda(\xi)\right\}, \quad where \quad \lambda(\xi) = e^{\frac{-4}{3}}\eta_\xi^{\frac{2}{3}}. \tag{21}$$

For $\xi < 0$ from (13) and (14) we have

$$W_1(u,\xi) = \left[\sin(\frac{\pi u}{2}) + O(u^{-1})\right]W_1(u,-\xi) + \left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right]W_2(u,-\xi), \tag{22}$$

$$W_2(u,\xi) = \left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right]W_1(u,-\xi) + \left[-\sin(\frac{\pi u}{2}) + O(u^{-1})\right]W_2(u,-\xi). \tag{23}$$

By derivations of $W_1(u,\xi)$ and $W_2(u,\xi)$ and making use of (20), (21) we obtain the leading term of

$$\frac{\partial W_1(u,\xi)}{\partial\xi} = \left[sin(\frac{\pi u}{2}) + O(u^{-1})\right]W_1(u,-\xi)\left[\xi u + \sqrt{2u}\lambda(-\xi)\right] +$$
$$\left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right]W_2(u,-\xi)\left[\xi u + \sqrt{2u}\mu(\xi)\right]. \tag{24}$$

Similarly $\frac{\partial W_2(u,\xi)}{\partial\xi}$ can be written following form

$$\frac{\partial W_2(u,\xi)}{\partial\xi} = \left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right]\frac{\partial W_1(u,-\xi)}{\partial\xi} + \left[-sin(\frac{\pi u}{2}) + O(u^{-1})\right]\frac{\partial W_2(u,-\xi)}{\partial\xi}. \tag{25}$$

If, $\xi = -b$, then we calculate,

$$\frac{\partial W_1(u,-b)}{\partial\xi} = \left[sin(\frac{\pi u}{2}) + O(u^{-1})\right]W_1(u,b)\left[\sqrt{2u}\lambda(b) - bu\right] +$$

$$\left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right] W_2(u,b)\left[\sqrt{2u}\mu(b) - bu\right], \tag{26}$$

$$\frac{\partial W_2(u,-b)}{\partial \xi} = \left[\cos(\frac{\pi u}{2}) + O(u^{-1})\right] W_2(u,b)\left[\sqrt{2u}\lambda(b) - bu\right] +$$
$$\left[-sin(\frac{\pi u}{2}) + O(u^{-1})\right] W_1(u,b)\left[\sqrt{2u}\lambda(b) - bu\right]. \tag{27}$$

3 Asymptotic of eigenvalues for case a=-b with Neumann conditions

The eigenvalues of the operator L with Dirichlet boundary conditions and one transition point were investigated in [5]. In this section we will study distribution of the eigenvalues of operator L with boundary conditions $W'(a) = W'(b) = 0$. Note that, the positive distribution of the Neumann eigenvalues has been studied [1] in one and two turning points case. In this case the eigenvalues of equation(1) are the zeros of $\Delta_n(u) = 0$ where

$$\Delta_n(u_n) = \begin{vmatrix} W_1'(u,b) & W_2'(u,b) \\ W_1'(u,-b) & W_2'(u,-b) \end{vmatrix}. \tag{28}$$

The product of $\Delta_n(u_n)$ can be written in the form

$$\frac{\partial W_1(u,b)}{\partial \xi} \times \frac{\partial W_2(u,-b)}{\partial \xi} = W_1(u,b)\left\{bu - \sqrt{2u}\lambda(b)\right\} \tag{29}$$

$$\times \left[cos(\frac{\pi u}{2}) + O(u^{-1})\right] W_2(u,b)\left\{\sqrt{2u}\mu(b) - bu\right\} + \left[-sin\frac{\pi u}{2} + O(u^{-1})\right] W_1(u,b)\left\{\sqrt{2u}\lambda(b) - bu\right\}$$
$$= \left[cos(\frac{\pi u}{2}) + O(u^{-1})\right]\left[W_1(u,b)W_2(u,b)\left(bu\sqrt{2u}\mu(b) + bu\sqrt{2u}\lambda(b) - b^2u^2 - 2u\right)\right]$$
$$+ \left[-sin(\frac{\pi u}{2}) + O(u^{-1})\right]\left[W_1^2(u,b)\left(2bu\sqrt{2u}\lambda(b) - u^2b^2 - 2u\lambda^2(b)\right)\right].$$

The following term can be treated analogously

$$\frac{\partial W_1(u,-b)}{\partial \xi} \times \frac{\partial W_2(u,b)}{\partial \xi} = W_2(u,b)\left(bu - \sqrt{2u}\mu(b)\right) \tag{30}$$

$$\times \left[-sin(\frac{\pi u}{2}) + O(u^{-1})\right] W_1(u,b)\left\{\sqrt{2u}\lambda(b) - bu\right\} + [cos(\frac{\pi u}{2}) + O(u^{-1})]W_2(u,b)\left\{\sqrt{2u}\mu(b) - bu\right\}.$$

If we multiply $W_2(u,b)\{bu - \sqrt{2u}\mu(b)\}$ then (30) can be rewritten as follows

$$\frac{\partial W_1(u,-b)}{\partial \xi} \times \frac{\partial W_2(u,b)}{\partial \xi} = \left[-\sin(\frac{\pi u}{2}) + O(u^{-1})\right]\left[W_1(u,b))W_2(u,b)\left(bu\sqrt{2u}\mu(b) - b^2u^2 - 2u\right.\right. \tag{31}$$

$$+bu\sqrt{2u}\lambda(b)\Big)\Big] + \Big[\cos(\frac{\pi u}{2}) + O(u^{-1})\Big] W_2^2(u,b) \Big[2bu\sqrt{2u}\mu(b) - 2u\mu^2(b) - b^2u^2\Big].$$

For summarize and distinguish we suppose

$$A_1 = b^2u^2 - bu\sqrt{2u}\mu(b) + 2u - bu\sqrt{2u}\lambda(b), \tag{32}$$

$$A_2 = b^2u^2 - 2bu\sqrt{2u}\mu(b) + 2u\mu^2(b), \qquad A_3 = b^2u^2 - 2bu\sqrt{2u}\lambda(b) + 2u\lambda^2(b)$$

also M_1 and M_2 are defined in [4] in following form

$$M_1 = \sum_{s=0}^{n} u_s(\frac{2}{3}u\eta^{\frac{3}{2}})^{-s} + O(u^{-n-1}), \qquad M_2 = \sum_{s=0}^{n}(-1)^s u_s(\frac{2}{3}u\eta^{\frac{3}{2}})^{-s} + O(u^{-n-1}).$$

We know that the left sides of (29) and (31) are equal. Using (32) the following formula are valid

$$\Big[\cos(\frac{\pi u}{2}) + O(u^{-1})\Big] \Big[W_2^2(u,b)A_2 - W_1(u,b)W_2(u,b)A_1\Big] = \tag{33}$$

$$\Big[\sin(\frac{\pi u}{2}) + O(u^{-1})\Big] \Big[(W_1)^2(u,b)A_3 - W_1(u,b)W_2(u,b)A_1\Big].$$

By using (29) and let us suppose

$$\theta = \frac{4}{3}u\eta_b^{\frac{3}{2}}, \qquad L = \frac{1}{2}\pi^{-\frac{1}{2}}u^{-\frac{1}{2}}\Gamma(\frac{1}{2} + \frac{u}{2})(b^2-1)^{-\frac{1}{2}},$$

we can write

$$W_1(u,b)W_2(u,b) = LM_1M_2K_1K_2, \;\; W_1^2(u,b) = Le^{-\theta}M_2^2K_1^2A_3, \;\; W_2^2(u,b) = Le^{\theta}M_1^2K_2^2A_2,$$

so by using(33) we have

$$\frac{\sin(\frac{\pi u}{2}) + O(u^{-1})}{\cos(\frac{\pi u}{2}) + O(u^{-1})} = \frac{W_2^2(u,b)e^{\theta}A_2 - W_1(u,b)W_2(u,b)A_1}{W_1^2(u,b)e^{-\theta}A_3 - W_1(u,b)W_2(u,b)A_1}. \tag{34}$$

Hence by using the form of $W_1(u,b)$ and $W_2(u,b)$ and substituting in (34) we can write

$$\frac{\sin(\frac{\pi u}{2})(1 + O(u^{-1}))}{\cos(\frac{\pi u}{2})(1 + O(u^{-1}))} = \frac{M_1^2K_2^2e^{\theta}A_2 - M_1M_2K_1K_2A_1}{M_2^2K_1^2e^{-\theta}A_3 - M_1M_2K_1K_2A_1}. \tag{35}$$

By dividing the right hand of (35) to $M_2M_1K_1K_2A_1$ we derive

$$\tan(\frac{\pi u}{2}) = \frac{-1 + (M_1K_2A_2)(M_2K_1A_1)^{-1}e^{\theta}}{-1 + (M_2K_1A_3)(M_1K_2A_1)^{-1}e^{-\theta}}(1 + O(u^{-1})). \tag{36}$$

$$\tan(\frac{\pi u}{2}) = -1 + (M_1K_2A_2)(M_2K_1A_1)^{-1}e^{\theta}(1 + O(u^{-1})).$$

Now we must obtain, M_2K_1, M_1K_2, $\frac{M_2K_1}{M_1K_2}$, $\frac{A_1}{A_2}$, $\frac{M_2K_1A_1}{M_1K_2A_2}$

$$M_1K_2 = u_0A_0+(\frac{3}{2}A_0u_1\eta^{-\frac{3}{2}}-u_0B_0b)\frac{1}{u}+\frac{\sqrt{2}u_0B_0}{u\sqrt{u}}+(u_0A_1-\frac{3}{2}u_1bB_0\eta^{-\frac{3}{2}}+\frac{9}{4}A_0u_2\eta^{-3})\frac{1}{u^2}+O(u^{-3}), \quad (37)$$

$$M_2K_1 = u_0A_0+(u_0B_0b-\frac{3}{2}A_0u_1\eta^{-\frac{3}{2}})\frac{1}{u}-\frac{\sqrt{2}u_0B_0}{u\sqrt{u}}+(u_0A_1-\frac{3}{2}u_1bB_0\eta^{-\frac{3}{2}}+\frac{9}{4}A_0u_2\eta^{-3})\frac{1}{u^2}+O(u^{-3}). \quad (38)$$

If we divide M_2K_1 by M_1K_2 we obtain

$$\frac{M_2K_1}{M_1K_2} = 1+\frac{2}{u_0A_0}\left(u_0B_0b-3A_0u_1\eta_b^{-\frac{3}{2}}\right)\frac{1}{u}+\frac{2\sqrt{2}B_0}{A_0}\times\frac{1}{u\sqrt{u}}+O(u^{-2}). \quad (39)$$

Now by dividing A_1 to A_2 we have, $\frac{A_1}{A_2} = 1+\frac{2\sqrt{2}}{b\sqrt{u}}\sinh(\frac{4}{3}\eta_b^{\frac{2}{3}})+O(u^{-1})$. If we multiply both sides of (39) by $\frac{A_1}{A_2}$ we determine

$$\frac{M_2K_1A_1}{M_1K_2A_2} = 1+\frac{2(u_0B_0b-3A_0u_1\eta_b^{-\frac{3}{2}})}{u_0A_0u}-\frac{2\sqrt{2}B_0}{A_0u^{\frac{3}{2}}}+\frac{2\sqrt{2}}{b\sqrt{u}}\sinh(\frac{4}{3}\eta_b^{\frac{2}{3}})+ \quad (40)$$

$$\frac{4\sqrt{2}}{bu_0A_0u\sqrt{u}}\sinh(\frac{4}{3}\eta_b^{\frac{2}{3}})(u_0B_0b-3A_0u_1\eta_b^{-\frac{3}{2}})+O(u^{-2}).$$

Therefore by using (36) we can write

$$\tan(\frac{\pi u}{2}) = e^{\theta}(-\frac{M_1K_2A_2}{M_2K_1A_1}+O(e^{-\theta})). \quad (41)$$

If we suppose,

$$x = -\frac{M_1K_2A_2}{M_2K_1A_1}e^{\theta}(1+O(e^{-\theta}))$$

then, $\frac{1}{x} = -\frac{M_2K_1A_1}{M_1K_2A_2}e^{-\theta}(1+O(e^{-\theta}))$, $\tan(\frac{\pi u}{2}) = x$. So we can write, $\frac{\pi u}{2} = n\pi+\arctan x$. For $|x|>1$, we know that,

$$\arctan(x) = \frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\ldots = \frac{\pi}{2}-\frac{1}{x}[1-\frac{1}{3x^2}+\frac{1}{5x^4}-\ldots], |x|>1. \quad (42)$$

Now by using the equation (42) for $x \longrightarrow \infty$, one finds that

$$\frac{\pi u}{2} = n\pi+\frac{\pi}{2}+O(\frac{1}{x}) \Longrightarrow \qquad u = \frac{n\pi+\frac{\pi}{2}}{\frac{\pi}{2}}+O(\frac{1}{x}).$$

We know that, $\frac{\pi}{2} = \int_{-1}^{1}(1-\tau^2)^{\frac{1}{2}}d\tau$, at last, we obtained

$$u_n = \frac{n\pi+\frac{\pi}{2}}{\int_{-1}^{1}(1-\tau^2)^{\frac{1}{2}}d\tau}+O(\frac{1}{x}). \quad (43)$$

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