Journal of mathematics and computer science 7 (2013), 258-265

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Difference Equations and SBEC Optimal Codes

P. K. Das

Department of Mathematics, Shivaji College (University of Delhi), Raja Garden, Delhi-110027, India

pankaj4thapril@yahoo.co.in

Article history:
Received April 2013
Accepted May 2013
Available online June 2013

Abstract

This paper studies the patterns of the solutions of an equation on bounds for one kind of optimal codes that corrects all solid bursts of length *b* or less and no others. Difference equations that are satisfied by the solutions (namely the parameters-the code length and information digits of such codes) are obtained.

Keywords: Parity check digits, bounds, solid burst error, optimal codes, difference equation.

1. Introduction

In coding theory, perfect codes are considered to be of great importance. It was a constant research to search out perfect codes for several years. After a great deal of efforts, it was established by Tietavainen [10] and van Lint [11] that there are no perfect codes over prime power alphabet other than the Hamming codes [5] and the Golay's (23, 12, 7) binary code and his (11, 6, 5) ternary code [4]. Perfect codes were investigated in terms of random errors. After it was settled, researchers started to find codes that are not perfect in usual sense but they correct certain type of errors and no others. Such types of codes are called **optimal codes**. To the best of the author's knowledge, Sharma and Dass [9] were the first who studied such type of perfect codes.

Solid burst errors are common in many communication channels viz. semiconductor memory data [6], supercomputer storage system [1]. A solid burst may be defined as follows:

Definition 1: *A solid burst of length b is a vector with non-zero entries in some b consecutive positions and zero elsewhere.*

Das [2] has obtained a bound on the number of parity check digits for a linear code that corrects all solid bursts of length *b* or less as follows:

Theorem 1: *The number of parity check digits for a (n, k) linear code over GF(q) that corrects all solid bursts of length b or less is given by*

$$q^{n-k} \geq 1 + \sum_{i=1}^{b}(n-i+1)(q-1)^{i} \,. \tag{1}$$

Considering the inequality in (1) with equality, we get an equation which will give rise to one kind of optimal codes that will correct all solid bursts of length *b* or less and no others. The equation is follows:

$$q^{n-k} = 1 + \sum_{i=1}^{b}(n-i+1)(q-1)^{i} \,. \tag{2}$$

The optimal codes obtained from (2) may be termed as *b*-Solid burst error correcting (*b*-SBEC) optimal codes.

Difference equation is frequently used to refer to *any* recurrence relation. It has many applications like in biology, Economics, Communication. In digital signal processing, recurrence relations can model feedback in a system, where outputs at one time become inputs for future time. Very recently, Kazemi and Delavar [7] have used recurrence relation to provide a precise analysis of the *t*-th moment of the profile in random binary digital trees. So the study of difference equations/recurrence relations is of importance to many researchers. Dass et al. [3] explored the patterns of the known solutions obtained from the well-known Rao-Hamming sphere-packing bound with equality and obtained the difference equations which are satisfied by the solutions.

In this regards, this paper also analyzes the known parameters of *b*-SBEC optimal codes and gives an attempt to obtain difference equations that are satisfied by the solutions of the equation (2).

Among the solid burst errors, the first most probable errors are solid burst error of length 1 and 2. And the next probable error is of solid burst of length 3. Therefore the study is restricted to binary case and $b \leq 3$.

2. Difference equations and SBEC optimal codes

As mentioned in introduction, the *b*-SBEC optimal codes are obtained from the equation (2). In binary case, the equation (2) becomes

$$2^{n-k} = 1 + \sum_{i=1}^{b}(n-i+1)^{i} \,. \tag{3}$$

Now assigning different values of n-k in equation (3) for some fixed b, we will get different values of n and correspondingly k also.

Corresponding to the values of n-k = 1, 2, 3, . . . , suppose the values of n and k are n_1, n_2, n_3, …. and k_1, k_2, k_3, ….. respectively.

For any positive integer i, we define functions f and g as follows:

$$f(i) = n_i \,; \;\; g(i) = k_i.$$

2.1 For *b*=1

The equation (3) reduces to

$$2^{n-k} = 1+n.$$

which coincides with the well-known Rao-Hamming sphere-packing bound with equality [8]. The codes obtained are nothing but famous Hamming Codes.

Therefore all the results obtained by Dass et al. [3] automatically follow in this case.

2.2 For *b*=2

The equation (3) reduces to

$$2^{n-k} = 2n.$$

$$\Rightarrow \quad 2^{n-k-1} = n.$$

Now we assign values to n-k as 1, 2, 3, ……, and we get different values of n. The corresponding values of k will be calculated as n-(n-k). In the following table 1, we tabulate the values of n, k, Δ_n, Δ_n^2, Δ_n^3, . . . *and* Δ_k, Δ_k^2, Δ_k^3, . . . (restricted to first 10 values of n-k).

TABLE 1. PARITY CHECKS AND CORRESPONDING LENGTHS AND INFORMATION DIGITS

n-k	n	Δ_n	Δ_n^2	Δ_n^3	k	Δ_k	Δ_k^2	Δ_k^3 . . . .
1	1				0			
2	2	1			0	0		
3	4	2	1		1	1	1	
4	8	4	2	1	4	3	2	1
5	16	8	4	2	11	7	4	2
6	32	16	8	4	26	15	8	4
7	64	32	16	8	57	31	16	8
8	128	64	32	16	120	63	32	16
9	256	128	64	32	247	127	64	32
10	512	256	128	64	502	255	128	64

On the basis of the table the following observations are made (which are found to coincide with the observations of [3]):

1. Δ_n consists of various powers of 2.

2. Δ_n^l and Δ_k^l are identical for $l > 1$
3. Δ_n^l (Δ_k^l), $l > 1$ is a geometric progression with common ratio 2.
4. n and Δ_n^l are identical but for a shift of l positions.
5. n and Δ_k^l ($l > 1$) are also identical for a shift of l positions.

The above observations lead to the following results:

Theorem 2.1: For $b = 2$, the functions f and g satisfy the first order linear non homogenous difference equations:

$$\Delta f(i) = 2^{i-1} \text{ and } \Delta g(i) = 2^i - 1.$$

Proof: We have

$$2^{(n-k)_i - 1} = n_i$$

$$\text{Or,} \quad 2^{i-1} = n_i.$$

$$\begin{aligned}
\Delta f(i) &= f(i+1) - f(i) \\
&= n_{i+1} - n_i \\
&= 2^i - 2^{i-1} = 2^{i-1}. \qquad (*)
\end{aligned}$$

$$\begin{aligned}
\Delta g(i) &= g(i+1) - g(i) \\
&= k_{i+1} - k_i \\
&= \{n_{i+1} - (n-k)_{i+1}\} - \{n_i - (n-k)_i\} \\
&= \{n_{i+1} - (i+1)\} - \{n_i - i\}. \\
&= \{n_{i+1} - n_i\} - 1 \\
&= \Delta f(i) - 1 \\
&= 2^{i-1} - 1. \qquad \text{[from (*)]}
\end{aligned}$$

■

Theorem 2.2: For $b = 2$, the functions f and g are linearly independent solution of the l-order linear non homogenous difference equation:

$$\Delta^l \mathrm{X}(i) = 2^{i-1} \text{ for } l \geq 2.$$

Proof: We will prove the theorem by induction method.

For $l = 2$,

$$\Delta^2 f(i) = \Delta f(i+1) - \Delta f(i) = 2^i - 2^{i-1} = 2^{i-1}.$$

So, the result is true for $l = 2$.

Let the result is true for any l -1, i.e., $\Delta^{l-1} f(i) = 2^{i-1}$.

Now $\Delta^l f(i) = \Delta^{l-1} f(i+1) - \Delta^{l-1} f(i)$

$= 2^i - 2^{i-1} = 2^{i-1}$.

Also for $l = 2$,

$\Delta^2 g(i) = \Delta g(i+1) - \Delta g(i) = (2^i - 1) - (2^{i-1} - 1) = 2^{i-1}$.

So the result is true for $l = 2$.

Let the result is true for any l -1.

So, $\Delta^{l-1} g(i) = 2^{i-1}$.

Now $\Delta^l g(i) = \Delta^{l-1} g(i+1) - \Delta^{l-1} g(i)$

$= 2^i - 2^{i-1} = 2^{i-1}$.

Since $(n\text{-}k)_i = i \Rightarrow f(i) - g(i) = i$, f and g are linearly independent.

Hence the theorem. ■

2.3 For *b* = 3

The equation (3) reduces to

$$2^{n-k} = 3n-2.$$

This equation has no integer solution for odd values of n-k. The following table gives solutions for even values of n-k (restricted to the first 8 values) as obtained in case of b=2.

TABLE 2. PARITY CHECKS AND CORRESPONDING LENGTHS AND INFORMATION DIGITS

n-k	n	Δ_n	Δ_n^2	Δ_n^3	k	Δ_k	Δ_k^2	Δ_k^3 . . . .
2	2				0			
4	6	4			2	2		
6	22	16	12		16	14	12	
8	86	64	48	36	78	62	48	36
10	342	256	192	144	332	254	192	144
12	1366	1024	768	576	1354	1022	768	576
14	5462	4096	3072	2304	5448	4094	3072	2304
16	21846	16384	12288	9216	21830	16382	12288	9216

We observe the followings from the Table 2:

1. Δ_n consists of various powers of 4.
2. Δ_n^l and Δ_k^l are identical for $l > 1$
3. Δ_n^l (Δ_k^l), $l > 1$ is a geometric progression with common ratio 4.

These will lead to the following results:

Theorem 2.3: For *b* = 3, the functions *f* and *g* satisfy the first order linear non homogenous difference equations:

$$\Delta f(2i) = 2^{2i} \text{ and } \Delta g(2i) = 2^{2i} - 2.$$

Proof: We have

$$2^{(n-k)_i} = 3n_i - 2$$

$$\text{Or,} \quad \frac{2^i + 2}{3} = n_i.$$

$$\Delta f(2i) = f(2(i+1)) - f(2i)$$

$$= n_{2(i+1)} - n_{2i}$$

$$= \frac{2^{2(i+1)} + 2}{3} - \frac{2^{2i} + 2}{3}$$

$$= 2^{2i}.$$

Again,

$$\Delta g(2i) = g(2(i+1)) - g(2i)$$

$$= k_{2(i+1)} - k_{2i}$$

$$= \{n_{2(i+1)} - (n-k)_{2(i+1)}\} - \{n_{2i} - (n-k)_{2i}\}$$

$$= \left[\frac{2^{2(i+1)} + 2}{3} - 2(i+1)\right] - \left[\frac{2^{2i} + 2}{3} - 2i\right]$$

$$= 2^{2i} - 2.$$ ■

Theorem 2.4: For *b* = 3, the functions *f* and *g* are linearly independent solution of the *l*-order linear non homogenous difference equation:

$$\Delta^l X(2i) = 3^{l-1} \times 2^{2i} \text{ for } l \geq 2.$$

Proof: This theorem is also proved by induction method.

For $l = 2$,

$\Delta^2 f(2i) = \Delta f(2(i+1))$ - $\Delta f(2i) = 2^{2(i+1)} - 2^{2i} = 3\times 2^{2i}$.

So, the result is true for $l = 2$.

Let the result is true for any l -1, i.e., $\Delta^{l-1} f(2i) = 3^{l-2} \times 2^{2i}$.

Now $\Delta^l f(i) = \Delta^{l-1} f(2(i+1))$- $\Delta^{l-1} f(2i)$

$= 3^{l-2} \times 2^{2(i+1)} - 3^{l-2} \times 2^{2i}$

$= 3^{l-1} \times 2^{2i}$

Also for $l = 2$,

$\Delta^2 g(2i) = \Delta g(2(i+1))$ - $\Delta g(2i)$= $(2^{2(i+1)} - 2) - (2^{2i} - 2) = 3\times 2^{2i}$.

So the result is true for $l = 2$.

Following the above way, we can easily prove

$$\Delta^l g(2i) = 3^{l-1} \times 2^{2i}.$$

Also as theorem 2.2, *f* and *g* are linearly independent.

Hence the theorem. ■

Remark. For $b = 4$. The equation (3) reduces to

$$2^{n-k} = 4n-5.$$

This equation has no integer solution.

3. References

[1] J. Arlat and W.C. Carter, Implementation and Evaluation of a (b, k)-Adjacent Error Correcting / Detecting Scheme for Supercomputer Systems, IBM J. RES. DEVELOP., 28(2), 159-169 (**1984**).

[2] P.K. Das, Codes Detecting and Correcting Solid Burst Errors, Bulletin of Electrical Engineering and Informatics, 1(3), 225-232 (**2012**).

[3] P. Dass, L. Berardi and B.K. Dass, Difference equations and Rao-Hamming bound, Journal of Interdisciplinary Mathematics, 9(2), 397-403 (**2006**).

[4] M.J.E Golay, Notes on digital coding, Proc. IRE, 37, 657 (**1949**).

[5] R.W. Hamming, Error-detecting and error-correcting codes, Bell System Technical Journal, 29, 147-160 (**1950**).

[6] D.W. Jensen, Block code to efficiently correct adjacent data and/or check bit errors, Patent no: US 6604222 B1, Date of Patent Aug 5,*(www.google.com/patents/US6604222)*, (**2003**).

[7] R. Kazemi and S. Delavar, The Moments of the Profile in Random Binary Digital Trees, Journal of mathematics and computer science, 6(3), 176-190 **(2013)**

[8] W.W. Peterson and E.J. Weldon(Jr), *Error-Correcting Codes*, 2nd edition, The MIT Press, Mass, **1972**.

[9] B.D. Sharma and B.K. Dass, Adjacent error correcting binary perfect codes, J. Cybernetics, 7, 9-13 (**1977**).

[10] A. Tietavainen, On the nonexistence of perfect codes over finite fields, SIAM J. Appl. Math., 24, 88-96 (**1973**).

[11] J.H. van Lint, A survey of perfect codes, Rocky Mountain J. Math., 5, 199-226 (**1975**).