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Coupled Fixed Point Theorems for Symmetric (ϕ, ψ)-weakly Contractive Mappings in Ordered Partial Metric Spaces

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Abstract

We establish some coupled fixed point theorems for symmetric (ϕ, ψ)-weakly contractive mappings in ordered partial metric spaces. Some recent results of Berinde (Nonlinear Anal. 74 (2011), 7347-7355; Nonlinear Anal. 75 (2012), 3218-3228) and many others are extended and generalized to the class of ordered partial metric spaces.

Keywords: Coupled fixed point; Partial metric space; Contractions; Mixed monotone property.

2000 Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fixed point theory is an important and powerful tool to study the phenomenon of nonlinear analysis and is considered to be a bridge bond between pure and applied mathematics. This theory has its wide applications in economics, physical and life sciences. Problems in engineering where adaptive systems encounter with the concepts of convergence, optimal performance, and stability can be solved using fixed point theory. In 1994, Matthews [3] introduced the concept of partial metric space, that is a generalization of metric space in which each object does not necessarily have a zero distance from itself [3]. The motivation behind this concept was to obtain a modified version of Banach contraction principle, more generally to solve certain problems arising in computer science and in the theory of computation [3]. Works of Valero [4], Oltra and Valero [4] and Altun et. al. [5] provide some generalizations of the results in [3].

Presently, fixed point theory has been receiving much attention in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. Ran and Reurings [6] were the first to establish the results in this direction. These results were then extended by Nieto and Rodríguez-López [7] for non-decreasing mappings. Works noted in [8-14] are some examples in this direction. Bhaskar and Lakshmikantham [15] introduced the notion of coupled fixed points and proved some coupled fixed point theorems for a mapping satisfying mixed monotone property in partially ordered metric spaces. Berinde [1] presented true generalizations of the results of Bhaskar and Lakshmikantham [15]. Berinde [2], further presented a nice extension of his own work [1] and generalized the results noted in [15], and [16]. Presented work extend Berinde [1, 2] results to ordered partial metric spaces.

Let us recall the following definitions of mixed monotone mappings and coupled fixed point of a mapping.

Definition 1.1 ([15]). Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \qquad x_1 \leq x_2 \text{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \qquad y_1 \leq y_2 \text{ implies } F(x, y_1) \geq F(x, y_2)$$

Definition 1.2 ([15]). An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Matthews [3], introduced the definition of a partial metric space as follows.

Definition 1.3 ([3]). A partial metric on a nonempty set X is a function $p: X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

p1. $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

p2. $p(x, x) \leq p(x, y)$,

p3. $p(x, y) = p(y, x)$,

p4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Note that, if $p(x, y) = 0$, then $x = y$. But the self distance of any point need not be zero; hence the idea of generalizing metrics so that a metric on a nonempty set X is precisely a partial metric p on X such that $p(x, x) = 0$. An important example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $p(x, y) = \max\{x, y\}$. For more examples and some results on partial metric spaces, the reader is suggested to refer [6, 17-24, 5, 25-30, 4, 31].

It is worth mentioning that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X,p) converges to a point x ∈ X, with respect to τ_p, if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$. This will be denoted as $x_n \to x,\ n \to \infty$, or $\lim_{n\to\infty} x_n = x$.

If p is a partial metric on X, then the function $p^s: X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n,x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

It is clear that if the pair $(\mathbb{R}^+,p)$ is a partial metric space, where $p: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x,y) = \max\{x,y\}$, then the corresponding metric is

$$p^s(x,y) = 2\max\{x,y\} - x - y = |x-y|.$$

Interestingly, a limit of a sequence in a partial metric space need not be unique. Also, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n,y_n) \to p(x,y)$.

Defintion 1.4 ([3]). Let (X,p) be a partial metric space. Then,

(1) a sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite);
(2) the space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 1.5 ([3]). Let (X,p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).
(b) The space (X,p) is complete if and only if the metric space (X,p^s) is complete.

Definition 1.6. Let X be a nonempty set. Then $(X,\leq,p)$ is called an ordered partial metric space if

(i) $(X,\leq)$ is a partially ordered set, and
(ii) (X,p) is a partial metric space.

Let (X,p) be a partial metric. We endow the product space $X \times X$ with the partial metric υ defined as follows:

$$\text{for } (x,y),(u,v) \in X \times X, \qquad \upsilon\big((x,y),(u,v)\big) = p(x,u) + p(y,v).$$

A mapping $F: X \times X \to X$ is said to be continuous at $(x,y) \in X \times X$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $F\left(B_q\big((x,y),\delta\big)\right) \subseteq B_p(F(x,y),\varepsilon)$.

Lemma 1.7 ([31]). Let (X, p) be a partial metric space. Then the mapping $F: X \times X \to X$ is continuous if and only if given a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ and $(x, y) \in X \in X$ such that $v\big((x,y),(x,y)\big) = \lim_{n\to\infty} v\big((\mathrm{x},\mathrm{y}),(x_n,y_n)\big)$, then

$$p\big(F(x,y),F(x,y)\big) = \lim_{n\to\infty} p(F(x,y),F(x_n,y_n))$$

2. Main Results

Let Φ denote the class of functions $\phi : [0,\infty) \to [0,\infty)$ which satisfy

(ϕ1) ϕ is continuous and (strictly) increasing;

(ϕ2) $\phi(t) < t$ for all $t > 0$;

(ϕ3) $\phi(t+s) \le \phi(t) + \phi(s)$ for all $t, s \in [0,\infty)$.

Note that $\phi(t) = 0$ iff $t = 0$.

Let Ψ denote the class of functions $: [0,\infty) \to [0,\infty)$ which satisfy $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

Some examples of $\phi(t)$ are kt (where $k > 0$), $\frac{t}{t+1}, \frac{t}{t+2}$ and examples of $\psi(t)$ are kt (where $k > 0$), $\frac{\ln(2t+1)}{2}$.

Theorem 2.1. Let $(X, \le)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi\left(\frac{p\big(F(x,y),F(u,v)\big)+p\big(F(y,x),F(v,u)\big)}{2}\right) \le \phi\left(\frac{p(x,u)+p(y,v)}{2}\right) - \psi\left(\frac{p(x,u)+p(y,v)}{2}\right), \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \ge u$ and $y \le v$ (or $x \le u$ and $y \ge v$).

Suppose either

(a) F is continuous, or
(b) X has the following property:
  (i) if a non-decreasing sequence $\{x_n\} \to x\}$ in (x, p), then $x_n \le xx$, $\forall\, n$,
  (ii) if a non-increasing sequence $\{y_n\} \to y$ in (X, p), then y $\le y_n$, $\forall$ n.

If there exist two elements x_0, $y_0 \in$ X with

$$x_0 \le F(x_0, y_0) \text{ and } y_0 \ge F(y_0, x_0), \tag{2.2}$$

or

$$x_0 \ge F(x_0, y_0) \text{ and } y_0 \le F(y_0, x_0), \tag{2.3}$$

then there exist $x, y \in X$ such that

$x = F(x, y)$ and $y = F(y, x)$.

Proof. Without loss of generality, assume that there exist two elements $x_0, y_0 \in X$, $y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_0 \leq x_1$ and $y_0 \geq y_1$. Similarly, let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$. Since F has the mixed monotone property, then we have $x_1 \leq x_2$ and $y_1 \geq y_2$. Continuing in the same way, we can easily construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$ and

$$x_0 \leq x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots, \qquad y_0 \geq y_1 \geq y_2 \geq \cdots y_n \geq y_{n+1} \geq \cdots. \tag{2.4}$$

Now, we can apply inequality (2.1) with $(x, y) = (x_n, y_n)$ and $(u, v) = (x_{n+1}, y_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$. We get

$$\begin{aligned} \phi\left(\frac{p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})}{2}\right) &= \phi\left(\frac{p\big(F(x_n,y_n),F(x_{n+1},y_{n+1})\big)+p\big(F(y_n,x_n),F(y_{n+1},x_{n+1})\big)}{2}\right) \\ &\leq \phi\left(\frac{p(x_n,x_{n+1})+p(y_n,y_{n+1})}{2}\right) - \psi\left(\frac{p(x_n,x_{n+1})+p(y_n,y_{n+1})}{2}\right) \\ &\leq \phi\left(\frac{p(x_n,x_{n+1})+p(y_n,y_{n+1})}{2}\right), \end{aligned} \tag{2.5}$$

which, in turn, by condition (ϕ1) implies

$$\frac{p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})}{2} \leq \frac{p(x_n,x_{n+1})+p(y_n,y_{n+1})}{2},$$

showing that the sequence $\{\delta_n\}$ is non-increasing, where $\delta_n = \frac{p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})}{2}$. Therefore there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})}{2} = \delta. \tag{2.6}$$

We shall show that $\delta = 0$. Assume to the contrary, that is $\delta > 0$. Then by letting $n \to \infty$ in (2.5) we have

$$\begin{aligned} \phi(\delta) &= \lim_{n\to\infty} \phi(\delta_{n+1}) \\ &\leq \lim_{n\to\infty} \phi(\delta_n) - \lim_{n\to\infty} \psi(\delta_n) \\ &= \phi(\delta) - \lim_{\delta_n\to\delta^+} \psi(\delta_n) < \phi(\delta), \end{aligned}$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})}{2} = 0. \tag{2.7}$$

We now show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the partial metric space (X, p). For, we prove that

$$\lim_{n,m\to\infty} \frac{p(x_n,x_m)+p(y_n,y_m)}{2} = 0. \tag{2.8}$$

Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find the subsequences $\{x_{m(i)}\}$, $\{x_{n(i)}\}$ of $\{x_n\}$ and $\{y_{m(i)}\}$, $\{y_{n(i)}\}$ of $\{y_n\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad \frac{p(x_{m(i)},x_{n(i)})+p(y_{m(i)},y_{n(i)})}{2} \geq \varepsilon. \tag{2.9}$$

This means that

$$\frac{p(x_{m(i)},x_{n(i)-1})+p(y_{m(i)},y_{n(i)-1})}{2} < \varepsilon. \tag{2.10}$$

By p4 and (2.10), we have

$$\frac{p(x_{m(i)},x_{n(i)})+p(y_{m(i)},y_{n(i)})}{2}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(p(x_{m(i)},x_{m(i)+1})+p(x_{m(i)+1},x_{n(i)})-p(x_{m(i)+1},x_{m(i)+1})\big)\\+\big(p(y_{m(i)},y_{m(i)+1})+p(y_{m(i)+1},y_{n(i)})-p(y_{m(i)+1},y_{m(i)+1})\big)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(p(x_{m(i)},x_{m(i)+1})+p(x_{m(i)+1},x_{n(i)})\big)\\+\big(p(y_{m(i)},y_{m(i)+1})+p(y_{m(i)+1},y_{n(i)})\big)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(p(x_{m(i)},x_{m(i)+1})+p(x_{m(i)+1},x_{m(i)})+p(x_{m(i)},x_{n(i)})-p(x_{m(i)},x_{m(i)})\big)\\+\big(p(y_{m(i)},y_{m(i)+1})+p(y_{m(i)+1},y_{m(i)})+p(y_{m(i)},y_{n(i)})-p(y_{m(i)},y_{m(i)})\big)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(2p(x_{m(i)+1},x_{m(i)})+p(x_{m(i)},x_{n(i)})\big)\\+\big(2p(y_{m(i)+1},y_{m(i)})+p(y_{m(i)},y_{n(i)})\big)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(2p(x_{m(i)+1},x_{m(i)})+p(x_{m(i)},x_{n(i)-1})+p(x_{n(i)-1},x_{n(i)})-p(x_{n(i)-1},x_{n(i)-1})\big)\\+\big(2p(y_{m(i)+1},y_{m(i)})+p(y_{m(i)},y_{n(i)-1})+p(y_{n(i)-1},y_{n(i)})-p(y_{n(i)-1},y_{n(i)-1})\big)\end{array}\right\}$$

$$\leq \frac{1}{2}\left\{\begin{array}{l}\big(2p(x_{m(i)+1},x_{m(i)})+p(x_{m(i)},x_{n(i)-1})+p(x_{n(i)-1},x_{n(i)})\big)\\+\big(2p(y_{m(i)+1},y_{m(i)})+p(y_{m(i)},y_{n(i)-1})+p(y_{n(i)-1},y_{n(i)})\big)\end{array}\right\}$$

$$< 2\,\frac{p(x_{m(i)+1},x_{m(i)})+p(y_{m(i)+1},y_{m(i)})}{2} + \varepsilon + \frac{p(x_{n(i)-1},x_{n(i)})+p(y_{n(i)-1},y_{n(i)})}{2}. \tag{2.11}$$

Letting $i \to \infty$ in (2.11), using (2.7) and (2.9), we obtain

$$\lim_{i \to \infty} \frac{p(x_{m(i)},x_{n(i)})+p(y_{m(i)},y_{n(i)})}{2} = \varepsilon. \tag{2.12}$$

Also, we have

$$p(x_{m(i)}, x_{n(i)}) \leq p(x_{m(i)}, x_{n(i)-1}) + p(x_{n(i)-1}, x_{n(i)})$$

and

$$p(y_{m(i)}, y_{n(i)}) \leq p(y_{m(i)}, y_{n(i)-1}) + p(y_{n(i)-1}, y_{n(i)}).$$

Then, we obtain that

$$p(x_{m(i)}, x_{n(i)}) + p(y_{m(i)}, y_{n(i)}) \leq \{p(x_{m(i)}, x_{n(i)-1}) + p(y_{m(i)}, y_{n(i)-1})\} + \{p(x_{n(i)-1}, x_{n(i)}) + p(y_{n(i)-1}, y_{n(i)})\}. \tag{2.13}$$

Similarly, one can show that

$$p(x_{m(i)}, x_{n(i)-1}) + p(y_{m(i)}, y_{n(i)-1}) \leq \{p(x_{m(i)}, x_{n(i)}) + p(y_{m(i)}, y_{n(i)})\} + \{p(x_{n(i)}, x_{n(i)-1}) + p(y_{n(i)}, y_{n(i)-1})\}. \tag{2.14}$$

Letting $i \to \infty$ in (2.13)-(2.14), and using (2.12), (2.7), we obtain that

$$\lim_{i \to \infty} \frac{p(x_{m(i)}, x_{n(i)-1}) + p(y_{m(i)}, y_{n(i)-1})}{2} = \varepsilon. \tag{2.15}$$

Since $x_{m(i)} \leq x_{n(i)-1}$ and $y_{m(i)} \geq y_{n(i)-1}$, we have

$$\phi\left(\frac{p(x_{n(i)}, x_{m(i)+1}) + p(y_{n(i)}, y_{m(i)+1})}{2}\right)$$

$$= \phi\left(\frac{p\left(F(x_{n(i)-1}, y_{n(i)-1}), F(x_{m(i)}, y_{m(i)})\right) + p\left(F(y_{n(i)-1}, x_{n(i)-1}), F(y_{m(i)}, x_{m(i)})\right)}{2}\right)$$

$$\leq \phi\left(\frac{p(x_{n(i)-1}, x_{m(i)}) + p(y_{n(i)-1}, y_{m(i)})}{2}\right) - \psi\left(\frac{p(x_{n(i)-1}, x_{m(i)}) + p(y_{n(i)-1}, y_{m(i)})}{2}\right).$$

Letting $i \to \infty$ in the above inequality, using (2.15) and the properties of ϕ and ψ, we get

$$\phi(\varepsilon) \leq \phi(\varepsilon) - \lim_{i \to \infty} \psi\left(\frac{p(x_{n(i)-1}, x_{m(i)}) + p(y_{n(i)-1}, y_{m(i)})}{2}\right) < \phi(\varepsilon),$$

a contradiction. Therefore (2.8) holds, and we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{2.16}$$

By (1.2), we have

$$p^s(x_n, x_m) \leq 2\, p(x_n, x_m) \quad \text{and} \quad p^s(y_n, y_m) \leq 2\, p(y_n, y_m). \tag{2.17}$$

Letting n, m $n, m \to \infty$ in (2.17) and using (2.16), we get

$$\lim_{n,m\to\infty} p^s(x_n, x_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} p^s(y_n, y_m) = 0. \tag{2.18}$$

Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (X, p^s). Since (X, p) is complete, it is also the case for (X, p^s). Then, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} p^s(x_n, x) = 0 \quad \text{and} \quad \lim_{n\to\infty} p^s(y_n, y) = 0. \tag{2.19}$$

On the other hand, we have

$$p^s(x_n, x) = 2p(x_n, x) - p(x_n, x_n) - p(x, x).$$

Letting $n \to \infty$ in the above equation, using (2.19) and (2.16), we get

$$\lim_{n\to\infty} p(x_n, x) = \frac{1}{2} p(x, x). \tag{2.20}$$

On the other hand, we have p(x, x) $p(x, x) \leq p(x, x_n)$ for all $n \in \mathbb{N}$. On letting $n \to \infty$, we get

$$p(x, x) \leq \lim_{n\to\infty} p(x, x_n). \tag{2.21}$$

Using (2.20) and (2.21), we get

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0.$$

Similarly, one can show that

$$\lim_{n\to\infty} p(y_n, y) = 0\ .$$

Therefore,

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(y_n, y) = p(y, y) = 0\ . \tag{2.22}$$

By p2, we have $0 \leq p(x_n, x_n) \leq p(x_n, x)$ and $p \leq p(y_n, y_n) \leq p(y_n, y)$ for all $n \in \mathbb{N}$. On letting $n \to \infty$ and using (2.22), we get that

$$\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x) = 0$$

$$\lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = p(y, y) = 0 \tag{2.23}$$

Now, we show that $x = F(x, y)$ and $y = F(y, x)$.

Suppose that the assumption (a) holds.

We follow the following steps.

Step I. We show that $p\big(F(x, y), F(x, y)\big) = 0$ and $p\big(F(y, x), F(y, x)\big) = 0$.

Since $x \leq x$ and $y \leq y$, we have

$$\phi\left(\frac{p\big(F(x,y),F(x,y)\big)+p\big(F(y,x),F(y,x)\big)}{2}\right) \leq \phi\left(\frac{p(x,x)+p(y,y)}{2}\right) - \psi\left(\frac{p(x,x)+p(y,y)}{2}\right)$$

$$= \phi(0) - \psi(0) = -\psi(0) \leq 0,$$

which implies $\frac{p(F(x,y),(x,y))+p(F(y,x),F(y,x))}{2} = 0$, and hence $p(F(x,y),(x,y)) = 0$ and $p(F(y,x),F(y,x)) = 0$.

Step II. We show that $\lim_{n\to\infty} p(x_{n+1},F(x,y)) = p(F(x,y),F(x,y))$ and $\lim_{n\to\infty} p(y_{n+1},F(y,x)) = p(F(y,x),F(y,x))$

We have $p(x_{n+1},F(x,y)) = p(\mathrm{F}(x_n,y_n),F(x,y))$. Since $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in (X,p), and F is continuous, by Lemma 1.7, we get $F(x_n,y_n) \to F(u,v)$ as $n \to \infty$ in (X,p); that is,

$$\lim_{n\to\infty} p(\mathrm{F}(x_n,y_n),F(x,y)) = p(F(x,y),F(x,y)) = 0.$$

Similarly, one can see that $\lim_{n\to\infty} p(\mathrm{F}(y_n,x_n),F(y,x)) = p(F(y,x),F(y,x)) = 0$.

Step III. We show that $x = F(x,y)$ and $y = F(y,x)$.

We have

$$p(x,F(x,y)) \le p(x,x_{n+1}) + p(x_{n+1},F(x,y)) - p(x_{n+1},F(x,y)) - p(x_{n+1},F(x,y))$$
$$\le p(x,x_{n+1}) + p(x_{n+1},F(x,y)).$$

Letting $n \to \infty$ in the above inequality, using (2.23) and Step II, we can obtain $p(u,F(u,v)) = 0$. Thus, we have $x = F(x,y)$. Similarly, we can show that $y = F(y,x)$.

Finally, suppose that the assumption (b) holds. By (2.4), (2.23) we have that $\{x_n\}$ is a non-decreasing sequence, $x_n \to x$ in (X,p) and $\{y_n\}$ is a non-increasing sequence, $y_n \to y$ in (X,p) as $n \to \infty$. Hence, by assumption (b), we have for all $n \ge 0$, that

$$x_n \le x \text{ and } y \le y_n . \tag{2.24}$$

By (2.1), we have

$$\phi\left(\frac{p(x_{n+1},F(x,y))+p(y_{n+1},F(y,x))}{2}\right) = \phi\left(\frac{p(\mathrm{F}(x_n,y_n),F(x,y))+p(\mathrm{F}(y_n,x_n),F(y,x))}{2}\right)$$
$$\le \phi\left(\frac{p(x_n,x)+p(y_n,y)}{2}\right) - \psi\left(\frac{p(x_n,x)+p(y_n,y)}{2}\right).$$

On letting $n \to \infty$ in the above inequality, using (2.23) and the properties of ϕ and ψ, we get

$$\lim_{n\to\infty} p(x_{n+1},F(x,y)) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(y_{n+1},F(y,x)) = 0. \tag{2.25}$$

On the other hand, we have

$$p(x,F(x,y)) \le p(x,x_{n+1}) + p(x_{n+1},F(x,y)) - p(x_{n+1},x_{n+1})$$
$$\le p(x,x_{n+1}) + p(x_{n+1},F(x,y)).$$

Letting $n \to \infty$ in the above inequality, using (2.23) and (2.25), we have $p(x, F(x, y)) = 0$; that is $x = F(x, y)$. Similarly, we have

$$p(y, F(y, x)) \leq p(y, y_{n+1}) + p(y_{n+1}, F(y, x)) - p(y_{n+1}, y_{n+1})$$
$$\leq p(y, y_{n+1}) + p(y_{n+1}, F(y, x)).$$

Letting $n \to \infty$ in the above inequality, using (2.23) and (2.25), we have $p(y, F(y, x)) = 0$; that is $y = F(y, x)$. Hence we proved that (x, y) is a coupled fixed point of the mapping F.

Example 2.2. Let $X = \mathbb{R}$**,** endowed with the partial metric given by $p(x, y) = \max\{x, y\}$ with the natural ordering of real numbers and define $F: X \times X \to X$ by

$$F(x, y) = \frac{x - y}{8}$$

for all $x, y \in$ X.

Obviously F has the mixed monotone property. Now, we show that F satisfies condition (2.1). Indeed, we have

$$p(F(x, y), F(u, v)) = \max\left\{\frac{|x - y|}{8}, \frac{|u - v|}{8}\right\} = \frac{1}{8}\max\{x - y, y - x, u - v, v - u\}$$

$$= \frac{1}{8}\max\{x, y, u, v\} \leq \frac{1}{8}\max\{x, u\} + \frac{1}{8}\max\{y, v\}.$$

Similarly, we have

$$p(F(y, x), F(v, u)) \leq \frac{1}{8}\max\{x, u\} + \frac{1}{8}\max\{y, v\}.$$

Then, by summing up the two inequalities, we obtain

$$p(F(x, y), F(u, v)) + p(F(y, x), F(v, u)) \leq \frac{p(x, u) + p(y, v)}{8} + \frac{p(x, u) + p(y, v)}{8},$$

that is

$$p(F(x, y), F(u, v)) + p(F(y, x), F(v, u)) \leq \frac{p(x, u) + p(y, v)}{2} - \frac{1}{2}\frac{p(x, u) + p(y, v)}{2}$$

and so condition (2.1) holds with $\phi(t) = t/2$ and $\psi(t) = 3t/8$. All the other hypotheses of Theorem 2.1 are easily satisfied and (0, 0) is a coupled fixed point of F.

Remark 2.3. Theorem 2.1 extends and generalizes Theorem 2 in [2].

Corollary 2.4. Let $(X,\leq)$ be a partially ordered set and p be a partial metric on X such that (X,p) is a complete partial metric space. Let $F: X\times X\to X$ be a mixed monotone mapping for which there exists $\psi_1\in\Psi$ such that for all $x,y,u,v\in X$ with $x\geq u,\ y\leq v$ (or $x\leq u,\ y\geq v$),

$$p\big(F(x,y),F(u,v)\big)+p\big(F(y,x),F(v,u)\big)\leq p(x,u)+p(y,v)-2\psi_1\left(\frac{p(x,u)+p(y,v)}{2}\right). \tag{2.26}$$

Suppose either

(a) F is continuous, or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in (X,p), then $x_n\leq x$, $\forall\, n$,
(ii) if a non-increasing sequence $\{y_n\}\to y$y in (X,p), then $y\leq y_n$, $\forall\, n$.

If there exist two elements $x_0,y_0\in X$ such that either (2.2) or (2.3) is satisfied, then there exist x, $y\in X$ such that

$$x=F(x,y)\quad\text{and}\quad y=F(y,x).$$

Proof. Note that if $\psi_1\in\Psi$, then for all $r>0$, $\psi_1\in\Psi$. Now divide (2.26) by 4 and take $(t)=\frac{1}{2}t$, $t\in[0,\infty)$, then condition (2.26) reduces to (2.1) with $\psi=2\psi_1$; and hence by Theorem 2.1 we obtain Corollary 2.4.

Corollary 2.5. Let $(X,\leq)$ be a partially ordered set and p be a partial metric on x such that (X,p) is a complete partial metric space. Let $F: X\times X\to X$ be a mixed monotone mapping and suppose that there exists some $k\in[0,1)$ such that for all $x,y,u,v\in X$ with $x\geq u$, $y\leq v$ (or $x\leq u, y\geq v$),

$$p\big(F(x,y),F(u,v)\big)+p\big(F(y,x),F(v,u)\big)\leq k\,[p(x,u)+p(y,v)]. \tag{2.27}$$

Suppose either

(a) F is continuous, or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\}\to x$ in (X,p), then $x_n\leq x$, $\forall\, n$,
(ii) if a non-increasing sequence $\{y_n\}\to y$ in (X,p), then $y\leq y_n$, $\forall\, n$.

If there exist two elements $x_0,y_0\in X$ such that either (2.2) or (2.3) is satisfied, then there exist $x,y\in X$ such that

$$x=F(x,y)\quad\text{and}\ y=F(y,x).$$

Proof. Taking $\varphi(t)=\frac{t}{2}$ and $\psi(t)=(1-k)\frac{t}{2}$, $0\leq k<1$ in Theorem 2.1, we obtain Corollary 2.4.

Remark 2.6. Corollary 2.5 extends and generalizes Theorem 3 in [1].

3. Uniqueness of Coupled Fixed Point

In this section we establish the uniqueness of coupled fixed point for our main result proved in Section 2.

If $(X, \leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order

$$\text{for} \quad (x, y), (u, v) \in X \times X, \quad (x, y) \leq (u, v) \Leftrightarrow x \leq u, u \geq v.$$

Analogously, $(x, y) \geq (u, v) \Leftrightarrow x \geq u, u \leq v$.

Then, we say that (x, y) and (y, x) are comparable if $(x, y) \leq (u, v)$ or $(x, y) \geq (u, v)$.

Theorem 3.1. In addition to the hypotheses of Theorem 2.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$, there exists a $(u, v) \in X \times X$ that is comparable to (x, y) and (x^*, y^*). Then F has a unique coupled fixed point.

Proof. From Theorem 2.1, the set of coupled fixed points of F is nonempty. Assume that (x, y) and (x^*, y^*) are two coupled fixed points of F, then we shall show that

$$p(x, x^*) = 0 \quad \text{and} \quad p(y, y^*) = 0.$$

By assumption, there exists a $(u, v) \in X \times X$ that is comparable to (x, y) and (x^*, y^*). We define the sequences $\{u_n\}$ and $\{v_n\}$ as follows:

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = F(u_n, v_n), \quad v_n = F(v_n, u_n), \quad n \geq 0.$$

Since (u, v) is comparable to (x, y), we may assume $(x, y) \geq (u, v) = (u_0, v_0)$. Following the proof of Theorem 2.1 we obtain inductively

$$(x, y) \geq (u_n, v_n), \quad n \geq 0 \tag{3.1}$$

and therefore, by (2.1),

$$\phi\left(\frac{p(x, u_{n+1}) + p(y, v_{n+1})}{2}\right) = \phi\left(\frac{p(F(x,y), F(u_n, v_n)) + p(F(y,x), F(v_n, u_n))}{2}\right)$$

$$\leq \phi\left(\frac{p(x, u_n) + p(y, v_n)}{2}\right) - \psi\left(\frac{p(x, u_n) + p(y, v_n)}{2}\right), \tag{3.2}$$

which, by the non-negativity of ψ, implies

$$\phi\left(\frac{p(x, u_{n+1}) + p(y, v_{n+1})}{2}\right) \leq \phi\left(\frac{p(x, u_n) + p(y, v_n)}{2}\right).$$

Thus, by the monotonicity of ϕ, we obtain that the sequence $\{\alpha_n\}$ defined by

$$\alpha_n = \frac{p(x, u_n) + p(y, v_n)}{2}, \quad n \geq 0,$$

is non-increasing. Hence, there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} \alpha_n = \alpha$.

We shall show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Letting $n \to \infty$ in (3.2), we get

$$\phi(\alpha) \leq \phi(\alpha) - \lim_{n\to\infty} \psi(\alpha_n) = \phi(\alpha) - \lim_{\alpha_n \to \alpha} \psi(\alpha_n) < \phi(\alpha),$$

a contradiction. Thus $\alpha = 0$; that is,

$$\lim_{n\to\infty} \frac{p(x,u_n)+p(y,v_n)}{2} = 0,$$

which implies

$$\lim_{n\to\infty} p(x,u_n) = \lim_{n\to\infty} p(y,v_n) = 0.$$

Similarly, we obtain that

$$\lim_{n\to\infty} p(x^*,u_n) = \lim_{n\to\infty} p(y^*,v_n) = 0.$$

By p4,

$$p(x,x^*) \le p(x,u_n) + p(u_n,x^*) - p(u_n,u_n)$$
$$\le p(x,u_n) + p(u_n,x^*),$$

on letting $n \to \infty$, we obtain $p(x,x^*) = 0$.

Similarly, we can obtain $p(y,y^*) = 0$. Hence, $x = x^*$ and $y = y^*$. This completes our proof.

Theorem 3.2. In addition to the hypotheses of Theorem 2.1, suppose that $x_0, y_0 \in X$ are comparable. Then F has a unique fixed point; that is, there exists $x \in X$ such that $F(x,x) = x$.

Proof. We claim that if (x,y) is a coupled fixed point of F, then $x = y$. Suppose the contrary $x \neq y$. By Theorem 2.1, without loss of generality, we assume that

$$x_0 \le F(x_0,y_0) \text{ and } y_0 \ge F(y_0,x_0).$$

Since x_0, y_0 are comparable, we have $x_0 \le y_0$ or $x_0 \ge y_0$. We assume $x_0 \ge y_0$. Then, by mixed monotone property of F, we have

$$x_1 = F(x_0,y_0) \ge F(y_0,x_0) = y_1,$$

and, hence, by making use of induction, we can get

$$x_n \ge y_n, \quad n \ge 0.$$

Also,

$$\lim_{n\to\infty} p(x,x_n) = 0 \text{ and } \lim_{n\to\infty} p(y,y_n) = 0.$$

Repeatedly applying p4,

$$p(x,y) \le p(x,x_{n+1}) + p(x_{n+1},y) - p(x_{n+1},x_{n+1})$$
$$\le p(x,x_{n+1}) + p(x_{n+1},y)$$
$$\le p(x,x_{n+1}) + p(x_{n+1},y_{n+1}) + p(y_{n+1},y) - p(y_{n+1},y_{n+1})$$
$$\le p(x,x_{n+1}) + p(x_{n+1},y_{n+1}) + p(y_{n+1},y)$$
$$= p(x,x_{n+1}) + p\big(F(x_n,y_n),F(y_n,x_n)\big) + p(y_{n+1},y),$$

then by monotonicity of ϕ and property (ϕ3), we get

$$\phi(p(x,y)) \le \phi\left(p(x,x_{n+1}) + p(F(x_n,y_n),F(y_n,x_n)) + p(y_{n+1},y)\right)$$

$$\le \phi(p(x,x_{n+1})) + \phi\left(p(F(x_n,y_n),F(y_n,x_n))\right) + \phi(p(y_{n+1},y))$$

$$\le \phi(p(x,x_{n+1})) + \phi(p(x_n,y_n)) - \psi(p(x_n,y_n)) + \phi(p(y_{n+1},y)),$$

on letting $n \to \infty$, using the properties of ϕ and ψ, we obtain

$$\phi(p(\mathrm{x},\mathrm{y})) \le \phi(0) + \phi(0) - \lim_{n\to\infty} \psi(p(x_n,y_n)) + \phi(0)$$

$$= -\lim_{n\to\infty} \psi(\mathrm{p}(x_n,y_n)).$$

We consider the following two cases:

Case I. If $\lim_{n\to\infty} p(x_n,y_n) > 0$, then $\lim_{n\to\infty} \psi(p(x_n,y_n)) > 0$, so that $\phi(p(x,y)) < 0$, that is a contradiction.

Case II. If $\lim_{n\to\infty} \mathrm{p}(x_n,y_n) = 0$, then $\lim_{n\to\infty} \psi(\mathrm{p}(x_n,y_n)) = 0$, so that $\phi(\mathrm{p}(\mathrm{x},\mathrm{y})) \le 0$.

Subcase I. If $\phi(p(x,y)) < 0$, we have a contradiction.

Subcase II. If $\phi(p(x,y)) = 0$, then $p(x,y) = 0$, so that $x = y$. Again, a contradiction to the assumption that $x \neq y$.

Hence, in all the cases we obtain contradiction, so our assumption that $x \neq y$ is wrong. Thus, we obtain $x = y$.

4. An Application

As consequences of our Theorem 2.1, we can obtain the following result for mappings with the mixed monotone property satisfying a contraction of integral type.

Firstly, denote by Λ the set of functions $\mu: [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(a1) μ is a Lebesgue-integrable function on each compact of $[0,\infty)$;

(a2) for every $\varepsilon > 0$, we have $\int_0^\varepsilon \mu(t)dt > 0$.

Then, we have

Theorem 4.1. Let $(X,\le)$ be a partially ordered set and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that, for all $x,y,u,v \in X$ with $x \ge u$ and $y \le v$ (or $x \le u$ and $y \ge v$), we have

$$\int_0^{\frac{p(F(x,y),F(u,v))+p(F(y,x),F(v,u))}{2}} \mu_1(t)dt \leq \int_0^{\frac{p(x,u)+p(y,v)}{2}} \mu_1(t)dt - \int_0^{\frac{p(x,u)+p(y,v)}{2}} \mu_2(t)dt \tag{4.1}$$

where $\mu_1, \mu_2 \in \Lambda$.

Suppose either

(a) F is continuous, or
(b) X has the following property:
  (i) if a non-decreasing sequence $\{x_n\} \to x$ in (X,p), then $x_n \leq x$, $\forall\, n$,
  (ii) if a non-increasing sequence $\{y_n\} \to y$ in (X,p), then $y \leq y_n$, $\forall\, n$.

If there exist two elements $x_0, y_0 \in X$ such that either (2.2) or (2.3) is satisfied, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. Clearly, the function $s \mapsto \int_0^s \mu_i\,(t)dt$ (for $i = 1,2$) defined in $[0,\infty)$ is in Φ and in Ψ. Therefore, the assertions follow trivially by Theorem 2.1.

Remark 4.2. Results analogous to Theorem 4.1 can be obtained using Corollaries 2.4 and 2.5.

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