Journal of mathematics and computer science 8 (2014), 93-97

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Convergence of Common Fixed Point Theorems in Fuzzy Metric Spaces

Virendra Singh Chouhan
Department of Mathematics
Lovely Professional University,
Jalandhar (Punjab), India
Email-darbarvsingh@yahoo.com

A. Ganguly
Dept. of applied Mathematics
& Computational Science,
S. G. S. I.T.S,
Indore (M.P.), India

Article history:
Received May 2013
Accepted June 2013
Available online June 2013

Abstract

Fixed point theorems in a fuzzy metric space are proved by considering a contractive condition for a triplet of mappings and this is the totally a new approach for obtaining the fixed point .

Key words:- Fixed point, quasi-contraction, fuzzy metric space, Cauchy sequence.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The notion of fuzzy set was introduced by Zadeh [9]. It was developed extensively by many authors and used in various fields. In this paper we deal with the fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [3].The most interesting references in this direction are Chang [1], Cho [2], Grabiec [4], and Kaleva [5]. In the present paper, first we prove a common fixed point theorem. Then we study the relationship between the convergence of three sequences of mappings and the convergence of their common fixed points.

2. Preliminaries

Definition 2.1[8]. A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with the unit 1 such that $a*b \leq c*d$ and whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0, 1]$.

Definition 2.2[6]. The 3-tuple $(X, M, *)$ is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ a continuous t-norm and M is a fuzzy set in $X \times X \times [0,\infty)$ satisfying the following conditions:

for all x, y, z $\in$ X and s, t > 0.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty] \rightarrow [0,1]$ is left continuous,

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a fuzzy metric space.

Example 2.3.[3]. Let (X, d) be a metric space. Define $a * b = \min\{a,b\}$ and $M(x,y,t) = \frac{t}{t + d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, $*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Lemma 2.4.[4]. For all $x, y \in X$, M(x, y, .) is a non decreasing function.

Definition 2.5[4]. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to be a Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$, such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$. The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each, $\varepsilon > 0$, $t > 0$, $n_0 \geq N$ such that $M(x_n, x, t) > 1- \varepsilon$ for all $n \geq n_0$.

A fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence in it converges to a point in it.

Remark 2.6. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. Let (X, M, $*$) be a fuzzy metric space with the following conditions

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 2.7[2]. Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with $t * t > t$ for all $t \in [0,1]$ and condition (FM-6). If there exists a number $k \in (0,1)$ such that

$$M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{x_n\}$ is a Cauchy sequence in X

Lemma 2.8[7]. If for all $x, y \in X$, $t > 0$ with positive number $k \in (0,1)$ and

$$M(x, y, kt) \geq M(x, y, t),$$

then x = y.

3. Main results

Theorem 3.1. Let (X, M, $*$) be a complete fuzzy metric space. Suppose that P, Q and T are mappings from X to itself such that

(a) PT = TP, QT = TQ,

(b) $P(X) \cup Q(X) \subseteq T(X)$,

(c) T is continuous, and

(d) The triplet (P, Q, T) is a quasi-contraction, i.e.,

$$M(Px, Qy, kt) \geq \min\{M(Tx, Ty, t), M(Px, Tx, t), M(Ty, Qy, t), M(Ty, Px, t), M(Tx, Qy, t)\}. \quad (1)$$

with $k \in (0,1)$, then P, Q and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point in X. we define sequence $\{y_n\}$ and $\{x_n\}$ such that

$y_{2n} = Tx_{2n} = Qx_{2n-1}$ and $y_{2n+1} = Tx_{2n+1} = Px_{2n}$, n = 1, 2, . . . This is always possible because of the condition (b).

Now taking $x = x_{2n}$ and $y = x_{2n+1}$ in (1) we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &= M(Px_{2n}, Qx_{2n+1}, kt) \\ &\geq \min\{M(Tx_{2n}, Tx_{2n+1}, t), M(Px_{2n}, Tx_{2n}, t), (Tx_{2n+1}, Qx_{2n+1}, t), \\ &\qquad M(Tx_{2n+1}, Px_{2n}, t), M(Tx_{2n}, Qx_{2n+1}, t)\}. \\ &= \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+1}, t), \\ &\qquad M(y_{2n}, y_{2n+2}, t)\}. \end{aligned}$$

which implies

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$$

In general

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t). \quad (2)$$

To prove that $\{y_n\}$ is a Cauchy sequence we prove by the method of induction that for all $n \geq n_0$, and for every $m \in N$,

$$M(y_n, y_{n+m}, t) > 1 - \lambda. \quad (3)$$

From (2) we have

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, \tfrac{t}{k}) \geq M(y_{n-2}, y_{n-1}, \tfrac{t}{k^2}) \geq - - - \geq M(y_0, y_1, \tfrac{t}{k^n}) \to 1 \text{ as } n \to \infty.$$

For $t > 0$, $\lambda \in (0,1)$, there exists $n_0 \in N$, such that

$$M(y_n, y_{n+1}, t) > 1 - \lambda. \quad (4)$$

Thus (3) is true for m = 1. Suppose (3) is true for all m then we will show that it is also true for m+1.

Using the definition of fuzzy metric space, (2) and (3), we have

$$M(y_n, y_{n+m+1}, t) \geq \min\{ M(y_n, y_{n+m}, \tfrac{t}{2}), M(y_{n+m}, y_{n+m+1}, \tfrac{t}{2}) \} > 1 - \lambda.$$

Hence (3) is true for m+1.

Thus $\{y_n\}$ is a Cauchy sequence. By completeness of (X, M, $*$), $\{y_n\}$ convergence to some point z in X.

Therefore $\{Tx_{2n}\}$,$\{Px_{2n}\}$and $\{ Qx_{2n-1} \}$also converge to z.

By continuity of T and the fact that PT = TP, it follows that

$$TTx_{2n} \to Tz \text{ and } PTx_{2n} = TPx_{2n} \to Tz.$$

Taking $x = Tx_{2n}$ and y = z in (1) we have

$$M(PTx_{2n}, Qz, kt) \geq \min\{M(TTx_{2n}, Tz, t), M(PTx_{2n}, TTx_{2n}, t), M(Tz, Qz, t), M(Tz, PTx_{2n}, t), M(TTx_{2n}, Qz, t)\}.$$

Taking limit $n \to \infty$ we have

$$M(Tz, Qz, kt) = \min\{1, 1, M(Tz, Qz, t), 1, M(Tz, Qz, t)\}$$

$$= M(Tz, Qz, t)$$

Therefore,

$$M(Tz, Qz, kt) \geq M(Tz, Qz, t),$$

which give Tz = Qz. Similarly Tz = Pz.

Again taking $x = x_{2n}$ and y = z in (1) we can show that

$$z = Qz = Tz = Pz.$$

Uniqueness:

Let w be another common fixed point of P, Q and T. Then we have

$$M(Pz, Qw, kt) \geq \min\{M(Tz, Tw, t), M(Pz,Tz, t), M(Tw, Qw, t), M(Tw, Pz, t), M(Tz, Qw, t)\}.$$

or $$M(z, w, kt) \geq M(z, w, t).$$

Thus z = w and hence z is a unique common fixed point of P, Q and T.

Next, we have the convergence theorem.

Theorem 3.2. Let $(X, M, *)$ be a complete fuzzy metric space. Let $\{P_n\}$, $\{Q_n\}$ and $\{T_n\}$ be sequences of mappings from X to itself such that the triplet $\{P_n, Q_n, T_n\}$ is a quasi- contraction. If P, Q and T: $X \rightarrow X$ are point wise limit of P_n, Q_n, and T_n respectively, and if $k_n \rightarrow k$ then (P, Q, T) is a quasi-contraction. Furthermore, the sequence of the unique common fixed point u_n of P_n, Q_n, and T_n, converges to the unique common fixed point u of P, Q and T.

Proof : For any $x, y \in X$, we have , for $x \neq y$,

$$\begin{aligned} M(Px, Qy, kt) &\geq M(Px, P_nx, t) * M(P_nx, Q_ny, t) * M(Q_ny, Qy, t) \\ &\geq \min\{M(T_nx, T_ny, t), M(P_nx, T_nx, t), M(T_ny, Q_ny, t), M(T_ny, P_nx, t), \\ &\qquad M(T_nx, Q_ny, t)\} * M(P_nx, Px, t) * M(Q_ny, Qy, t)\} \end{aligned}$$

Since $P_nx \rightarrow Px$, $Q_ny \rightarrow Qy$, $T_nx \rightarrow Tx$

and $T_ny \rightarrow Ty$ for $x \neq y$, as $n \rightarrow \infty$, and also $k_n \rightarrow k < 1$,

we have,

$$\begin{aligned} M(Px, Qy, kt) &\geq \min\{M(Tx, Ty, t), M(Px, Tx, t), M(Ty, Qy, t), M(Ty, Px, t), \\ &\qquad M(Tx, Qy, t)\} * M(Px, Px, t) * M(Qy, Qy, t)\} \\ &= \min\{M(Tx, Ty, t), M(Px, Tx, t), M(Ty, Qy, t), M(Ty, Px, t), \\ &\qquad M(Tx, Qy, t)\} * 1 * 1. \\ &= \min\{M(Tx, Ty, t), M(Px, Tx, t), M(Ty, Qy, t), M(Ty, Px, t), \\ &\qquad M(Tx, Qy, t)\} \end{aligned}$$

Hence we get (P, Q, T) is a quasi-contraction, and P, Q and T have the unique common fixed point u since X is complete.

Now suppose that u_n be the common fixed point of P_n, Q_n, and T_n for each n. Then we have

$$\begin{aligned} M(u_n, u, kt) &\geq M(u_n, Q_nu, t) * M(Q_nu, u, t) \\ &= M(P_nu_n, Q_nu, t) * M(Q_nu, u, t) \\ &\geq \min\{M(T_nu_n, T_nu, t), M(P_nu_n, T_nu_n, t), M(T_nu, Q_nu, t), M(T_nu, P_nu_n, t), \\ &\qquad M(T_nu_n, Q_nu, t)\} * M(Q_nu, u, t) \\ &\geq \min\{M(u_n, T_nu, t), 1, M(T_nu, Q_nu, t), M(T_nu, u_n, t), M(u_n, Q_nu, t)\} \\ &\qquad * M(Q_nu, u, t) \\ &= \min\{M(u_n, T_nu, t), M(T_nu, Q_nu, t), M(T_nu, u_n, t), M(u_n, Q_nu, t)\} \\ &\qquad * M(Q_nu, u, t) \end{aligned}$$

Now if $M(T_nu, Q_nu, t)$ is minimum then

$$\begin{aligned} M(u_n, u, kt) &\geq M(T_nu, Q_nu, t) * M(Q_nu, u, t) \\ &= M(T_nu, u, t) * M(u, u, t) \end{aligned}$$

Since $T_n \rightarrow T$ and $Q_n \rightarrow Q$, point wise it follows that, as $n \rightarrow \infty$,

$M(u_n, u, t) \rightarrow 1$, and therefore $u_n \rightarrow u$.

Now if $M(u_n, T_nu, t)$ is minimum, then

$$M(u_n, u, kt) \geq M(u_n, T_n u, t) * M(Q_n u, u, t)$$

so that, as $n \to \infty$, again we can get $u_n \to u$.
Similarly, for the last term in the bracket, we can prove that $u_n \to u$.

References

[1] S. S. Chang, Fixed point theorems for fuzzy mappings. *Fuzzy Sets and Systems* 17(2) 181-187, **(1985)**.

[2] Y. J. Cho, Fixed points in fuzzy metric spaces. *J. Fuzzy Math.* 5(4), 949-962 **(1997)**.

[3] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64, 395-399 **(1994)**.

[4] M. Grabiec, Fixed points in fuzzy metric spaces. *Fuzzy Sets and Systems* 27(3), 385-389 **(1988)**.

[5] O. Kaleva, The completion of fuzzy metric spaces. *J. Math. Anal. Appl.* 109(1), 194-198 **(1985)**.

[6] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika* 11, 326-334, **(1975)**.

[7] S. N. Mishra, N. Sharma & S. L. Singh, Common fixed points of maps on fuzzy metric spaces. *Internat. J. Math. Math. Sci.* 17(2), 253-258 **(1994)**.

[8] B. Schweizer & A. Sklar, Statistical metric spaces. *Pacific J. Math.* 10, 313-334 **(1960)**.

[9] L.A. Zadeh, Fuzzy sets, *Inform and Control* 8, 338-353 **(1965)**.