The Journal of Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .1 No.3 (2010) 160-166

Numerical Range-Preserving Linear Maps Between C^*- Algebras

A. TAGHAVI AND R. PARVINIANZADEH

Department of Mathematics, Faculty of Basic Sciences, University of Mazandaran, P. O. Box 47416-1468, Babolsar, Iran.

Taghavi@nit.ac.ir
r.parvinian@umz.ac.ir

Received: January 2010, Revised: March 2010
Online Publication: August 2010

Abstract

In this paper, we discuss numerical range-preserving maps between C^*-algebras. As applications, we characterize such maps in terms of Jordan $*-$isomorphisms on unital C^*-algebras.

Keywords: C^*-algebra, Jordan homomorphism, Linear Preserving, Numerical range.

2000 Mathematics Subject Classification: Primary 47D25; Secondary 47B49, 47A10.

1. Introduction

In 1951 Kadison obtained the following generalization of the Banach- Stone's theorem to arbitrary C^*-algebras [9].

Theorem 1.1. *(Kadison) Let A and B be unital C^*-algebras and let ϕ be a unital linear isometry from A onto B. Then there exist a uniquely determined unitary $u \in B$ and a uniquely determined Jordan $^*-$isomorphism $\varphi : A \to B$ such that $\phi(a) = u\varphi(a)$ for all $a \in A$.*

Here, a Jordan $^*-$isomorphism ϕ is a bijective linear mapping with the property that $\phi(a^2) = \phi(a)^2$ for all $a \in A$ and which is self-adjoint, i.e. preserves self-adjoint elements.

Let A and B be two unital C^*-algebras. We always denote by e the unit both A and B. A linear map $\phi : A \to B$ is called unital if $\phi(e) = e$ and is called a Jordan homomorphism if $\phi(a^2) = (\phi(a))^2$ for any $a \in A$. Likewise we say that ϕ is numerical range-preserving if $W(\phi(a)) = W(a)$ for every a in A. Where $W(a)$ denote the numerical range of $a \in A$. For C^*-algebras A and B, a C^*-isomorphism ϕ of A into B is an isomorphism such that $\phi(a^*) = \phi(a)^*$ for all a in A. The set $\sigma(a)$ and $r(a)$ will denote the spectrum and the spectral radius of $a \in A$, respectively.

If A be a (unital) C^*-subalgebra of a C^*-algebra B and a is in A. Then $\sigma_A(a) = \sigma_B(a)$. A linear map $\phi : A \to B$ is called a spectral isometry if $r(\phi(a)) = r(a)$ for every element $a \in A$.

A 35-years ago problem of Kaplansky [10] asked where every surjective spectrum preserving linear mapping between unital C^*-algebras has to be a Jordan isomorphism. A linear map ϕ from C^*-algebra A into C^*-algebra B is called spectrum-preserving if $\sigma(\phi(a)) = \sigma(a)$ for all $a \in A$. An important step forward was made by Aupetit [1] by establishing the result for von Neumann algebras.

Conjecture 1.2. Every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism.

Evidently this conjecture is harder than Kaplansky's problem, the point we wish to make here is that the statement provides analogue of Kadison's theorem and above conjecture. The aim of this paper is to solve it for numerical range-preserving linear maps between C^*-algebras, in fact, we obtain such maps are isometry on normal elements. We prove that every unital surjective numerical range-preserving linear mapping between unital C^*-algebras is a Jordan $^*-$isomorphism.

Throughout this paper H will be a fixed complex Hilbert space and $B(H)$ is the algebra of all bounded linear operators on complex Hilbert space H.

The numerical range of an operator $T \in B(H)$ is the subset of the complex numbers $\mathcal{C}$, given by

$$W(T) = \{\langle Tx, x\rangle : \quad x \in H, \|x\| = 1\}$$

and the numerical radius w(T) of an operator $T \in B(H)$ is given by

$$w(T) = sup\{|\lambda| : \lambda \in W(T)\}.$$

2. Main results

Suppose A is a C^*-algebra and S is a subset of A, we recall that the C^*-algebra generated by S, denoted by $A[S]$, is the smallest C^*-subalgebra of A containing S. If $S = \{a_1, a_2, ..., a_n\}$, we write $A[S] =$

$A[a_1, a_2, ..., a_n]$. In particular, $A[a]$ is the C^*-subalgebra generated by a. It is clear that if a be normal in a C^*-algebra A, then $A[a]$ is commutative. Moreover, an element $x \in A$ belongs to $A[a]$ if and only x can be approximated in norm by polynomials in a and a^*, (see [14]). For every C^*-algebra A there exists a Hilbert space H such that A is C^*-isomorphic to a C^*-subalgebra of $B(H)$, (see [14]).

Theorem 2.1. [11] *Let A and B be unital C^*-algebras and let ϕ be a unital spectral isometry from A onto B. If A or B is commutative then ϕ is a multiplicative isomorphism.*

We say that $a \in A$ is positive if $a = b^*b$ for some $b \in A$. write $a \geq 0$ if and only if a is positive.

Lemma 2.2. *Let A and B be C^*-algebras and let ϕ be a numerical range-preserving linear mapping from A into B. Then ϕ is self-adjoint and injective.*

Proof. Since A and B are both C^*-algebras, then A and B can be represented as a C^*-subalgebras of $B(H)$ and $B(K)$, respectively for some Hilbert spaces H and K. Hence every element in A is a operator on H and its image is a operator on K. Let $a \in A$ be positive, since $W(\phi(a)) = W(a)$, we obtain that $\phi(a) \geq 0$. Since every self-adjoin element is the difference of two positive ones, thus ϕ preserves self-adjoints. Therefore ϕ is self-adjoint.
Suppose $a \in A$ and $\phi(a) = 0$, then $W(\phi(a)) = 0$, so $W(a) = 0$. If a be self-adjoint then $\|a\| = 0$ (since for every self-adjoint operator in A, we have $\|a\| = r(a) = w(a)$), hence $a = 0$. For arbitrary $a \in A$, write $a = a_1 + ia_2$ with a_1 and a_2 self-adjoint, since ϕ is linear and self-adjoint, we obtain $\phi(a_1) = \phi(a_2) = 0$. Thus ϕ is injective. □

Theorem 2.3. *Let A and B be unital C^*-algebras and let ϕ be a numerical range-preserving unital linear mapping from A onto B. Then ϕ is a Jordan $^*-$isomorphism. Furthermore if B is prime, Then ϕ is either a C^*-isomorphism or a C^*-anti-isomorphism.*

Proof. By Lemma 2.2 ϕ is self-adjoint and injective. Let $a \in A$ be a self-adjoint element. Let $S = \{a, e\}$, then $A_1 = A[S]$ is a commutative C^*-algebra. Define a linear map $\phi_1 : A_1 \to \phi(A_1)$ by $\phi_1(x) = \phi(x)$ for every $x \in A_1$. If $\phi(A_1)$ is a C^*-subalgebra of B, then $r(\phi_1(x)) = r(x)$ for every $x \in A_1$ because $w(x) = r(x) = \|x\|$ for every x normal in A. By Theorem 2.1, ϕ_1 is a multiplicative isomorphism, so that $\phi(a^2) = \phi(a)^2$ for every self-adjoint $a \in A$. Otherwise let $B_1 = B[\phi(A_1)]$ be the C^*-algebra generated by $\phi(A_1)$ in B and define $\psi_1 : B_1 \to \phi^{-1}(B_1)$ by $\psi_1(y) = \phi^{-1}(y)$ for every $y \in B_1$. Since $a \in A$ is self-adjoint, an element $x \in A$ belongs to A_1 if and only if x can be approximated in norm by polynomials in a. Hence

$$r(\phi(p_n(a)) + \phi(a)) = r(\phi(p_n(a) + a)) =$$

$$r(p_n(a) + a) \leq r(p_n(a)) + r(a) = r(\phi(p_n(a))) + r(\phi(a)).$$

Since ϕ is self-adjoint, by [2, Theorem 5.2.2] $\phi(a) \in Z(B_1)$, therefore B_1 is commutative because every element in B_1 can approximate in norm by polynomials in $\phi(a)$. If $\phi^{-1}(B_1)$ is a C^*-subalgebra of A, then ψ_1 is a multiplicative isomorphism and hence ϕ_1 is a multiplicative isomorphism. Otherwise let $A_2 = A[\phi^{-1}(B_1)]$, it is clear that $A_1 \subseteq A_2$. Define $\phi_2 : A_2 \to \phi(A_2)$ by $\phi_2(x) = \phi(x)$ for every $x \in A_2$. If $\phi(A_1)$ wasn't a C^*-subalgebra of B, with continuing this process we obtain sequences $\{A_n\}$ and $\{B_n\}$ of commutative C^*-subalgebras A and B respectively, such that $A_1 = A[a, e]$, $B_1 = B[\phi(A_1)]$, $A_n = A[\phi^{-1}(B_{n-1})]$ and $B_n = B[\phi(A_n)]$ for every $n \geq 2$,

$$A_1 \subseteq A_2 \subseteq ... \subseteq A$$

and

$$B_1 \subseteq B_2 \subseteq ... \subseteq B.$$

Define $A' = \cup_{n \geq 1} A_n$ and $B' = \cup_{n \geq 1} B_n$. It is clear that A' and B' are commutative C^*-subalgebras of A and B respectively. The linear map $\phi' : A' \to B'$ by $\phi'(x) = \phi(x)$ for every $x \in A'$ is a restriction of ϕ, in addition to, ϕ' is unital and spectral isometry. ϕ' is surjective because if $b \in B'$ there is a positive integer number n such that $b \in B_n = B[\phi(A_n)]$. Since $\psi_n : B_n \to \phi^{-1}(B_n)$ then $\psi_n(b) = x \in \phi^{-1}(B_n) \subseteq$

A_{n+1}, it follows that $\phi^{-1}(b) = x \in A_{n+1}$, so $b = \phi(x) = \phi'(x)$. Hence ϕ' is surjective. By Theorem 2.1 ϕ' is Jordan isomorphism. Furthermore, $\phi'(a^2) = \phi'(a)^2$, it follows that $\phi(a^2) = \phi(a)^2$ for every self-adjoint $a \in A$. Replacing a by $a + b$, where both a and b are self-adjoint, we get $\phi(ab+ba) = \phi(a)\phi(b)+\phi(b)\phi(a)$. Since every $x \in A$ can be written in the form $x = a+ib$ with a and b self-adjoint, the last relation implies that $\phi(x^2) = \phi(x)^2$, that is, ϕ is Jordan $^*-$isomorphism.

The proof last part of theorem is obvious because it is well known (for example, see [6, pp. 47-51]) that if B is a prime ring, then every Jordan homomorphism ϕ from ring A onto B is either a homomorphism or an anti-homomorphism. This completes the proof. □

Corollary 2.4. *Let A be a unital C^*-algebra and B a unital prime C^*-algebra. Suppose ϕ be a numerical range-preserving unital linear mapping from A onto B. If A contains a left invertible element which is not invertible and its image is left invertible, then ϕ is a C^*-isomorphism.*

Proof. By Theorem 2.3, ϕ is either a C^*-isomorphism or a C^*-anti-isomorphism since B is prime. Let $a \in A$ be an element that is left invertible but not invertible and $b \in A$ be a left inverse of A. If ϕ is a C^*-anti-isomorphism, then $e = \phi(ba) = \phi(a)\phi(b)$, which implies that $\phi(a)$ is right invertible. Hence $\phi(a)$ is invertible and $\phi(e) = e = \phi(b)\phi(a) = \phi(ab)$. Since ϕ is injective (Lemma 2.2), we get $ab = e$. Thus a is invertible, a contradiction. Hence ϕ is a C^*-isomorphism.

□

Acknowledgements: This research is partially supported by the Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran.

References

[1] B. Aupetit, *Spectrum-preserving linear mapping between Banach algebras or Jordan-Banach algebras*, J. London. Math. Soc **62**(2000), 917–924.

[2] B. Aupetit, *A primer on spectral theory*, Springer, New York, 1991.

[3] B. Aupetit and H. du Toit Mouton, *Spectrum preserving linear mapping in Banach algebras*, Studia Math. **109**(1994), 91–100.

[4] M. Bresar and P. Semrl, *Linear maps preserving the spectral radius*, J. Funct. Anal. **142**(1996), 360–368.

[5] M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, and P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal **59**(1984), no 3, 462–469.

[6] I. N. Herstein, *Topic in ring theory*, springer, Berlin, 1991.

[7] A. A. Jafarian and A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal **66**(1986), 255–266.

[8] C. Jianlian, H. Jinchuan, *Linear maps between Banach algebras compressing certain spectral functions*, Roc. Moun. J. Math. **34**(2004), 565–584.

[9] R. V. Kadison, *Isometries of operator algebras*, Annals of Math. **54**(1951), 325–338.

[10] I. Kaplansky, *Algebraic and analytic aspects s of operator algebras* CBMS Regional Conference Series in Mathematics 1 (American Mathematical Socitey, Providence, RI, 1970).

[11] M. Mathieu, *Towards a non-selfadjoint version of Kadison' theorem*, Annales Mathematicae et Informaticae. **32** (2005), 87-94.

[12] M. Mathieu, *Spectrally Bounded Operators on Simple C^*-algebras*, Proc. Amer. Math. Soc. **132** (2003), 443-446.

[13] A. R. Sourour, *Invertibility preserving linear maps on $L(X)$* , Trans. Amer. Math. Soc. **348**(1996), 13–30.

[14] K. Zho, *An Introduction to operator algebra* CRC, New york, 1993.