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COINCIDENCE POINT THEOREM ASYMPTOTIC CONTRACTION MAPPING IN FUZZY METRIC SPACE

Binayak S. Choudhury[1], Krishnapada Das[2] and Pradyut Das[3]
Department of Mathematics
Bengal Engineering and Science University, Shibpur
P.O. : B. Garden, Shibpur
Howrah - 711103, West Bengal, INDIA
1. binayak12@yahoo.co.in
2. kestapm@yahoo.co.in
3. pradyutdas747@gmail.com

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Abstract
In this paper we introduce the concept of asymptotically g-contraction in fuzzy metric space. We prove some coincidence point results in fuzzy metric spaces using asymptotic contraction. We also support our results by an example.

Keywords: Fuzzy metric space, fixed point, asymptotically regular, g-contraction.

AMS Subject Classification : 47H25, 54H10

1. Introduction.

Contraction mappings in probabilistic and fuzzy metric spaces have considered by many authors. Seghal and Bharucha-Reid were the first to introduce contraction mapping principle in probabilistic metric space [15]. The result has been known as Seghal contraction. The structures of these spaces allow to extend the contraction mapping principle to these spaces in more than one inequivalent ways. One such concept is C-contraction which was originally introduced by Hicks in [10] and subsequently studied and generalized in several works like [12, 13, 14]. Fuzzy metric spaces were first defined by Kramosil and Michalek [11] by an extension of probabilistic metric spaces. George and Veeramani [5] modified the definition of fuzzy metric spaces given by Kramosil and Michalek. The motivation for such modified definition is to ensure that the topology induced by the fuzzy metric be Hausdorff. Some other works may be noted in [1, 4, 6, 9, 14, 16, and 17].

It is in the above mentioned space that we establish two coincidence point results for three mappings. For this purpose we consider fuzzy g-contraction and fuzzy asymptotically g-contraction, the later concept we have here. A supporting example is also given.

Definition 1.1 t-norms

A t-norm is a function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions for $a,b,c,d \in [0,1]$

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(c, d) \geq \Delta(a, b)$ whenever $c \geq a$ and $d \geq b$,

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Examples of t-norm.

1. Minimum t-norm (T_M): $T_M(x, y) = \min(x,y)$.
2. Product t-norm (T_P): $T_P(x, y) = x.y$.
3. Lukasiewicz t-norm (T_L) : $T_L(x,y) = \max(x + y - 1, 0)$.

Definition 1.2 Fuzzy Metric Space (Kramosil and Michalek) [11]

The 3-tuple (X, M, Δ) is said to be a fuzzy metric space if X is an arbitrary set, Δ is a t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

i) $M(x, y, 0) = 0$,

ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x=y$,

iii) $M(x, y, t) = M(y, x, t)$,

iv) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$,

v) $M(x, y, .) : [0,\infty) \to [0,1]$ is left continuous for $x,y,z \in X$ and $t,s > 0$.

Definition 1.3 Fuzzy Metric Space (George and Veeramani) [5]

The 3-tuple (X, M, Δ) is said to be a fuzzy metric space if X is an arbitrary set, Δ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions:

i) $M(x, y, t) > 0$,

ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x=y$,

iii) $M(x, y, t) = M(y, x, t)$,
iv) $M(x, z, t + s) \geq \Delta (M(x, y, t), M(y, z, s))$,
v) $M(x, y, .) : [0,\infty) \to [0,1]$ is continuous for $x,y,z \in X$ and $t,s > 0$.

Lemma 1.4 [8] Let (X, M,Δ) be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x,y \in X$.

Proof. If possible let $M(x, y, t) > M(x, y, s)$ for some $s, t > 0$ with $s > t$.
Then we have,
$$\begin{aligned} M(x, y, t) &> M(x, y, s) \\ &\geq \Delta (M(x, y, t), M(y, y, s - t)) \\ &\geq \Delta (M(x, y, t), 1) \\ &= M(x, y, t), \text{ which is a contractdion.} \end{aligned}$$
Hence $M(x, y, .)$ is nondecreasing.

Definition 1.5 [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, Δ) is said to converge to $x \in X$ if for all $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$.

Definition 1.6 [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M,Δ) **is** a Cauchy sequence if for each $\varepsilon > 0$, $t > 0$ there exits $n_0 \in \mathbf{N}$, the set of all natural numbers, such that
$$M (x_m, x_n, t) > 1 - \varepsilon \text{ for all } n, m \geq n_0.$$

Definition 1.7 [4] A fuzzy metric space (X, M,Δ) is said to be complete if every Cauchy sequence is convergent in it.

Definition 1.8 [4] Let (X, M,Δ) be a fuzzy metric space and $T: X \to X$ is a self mapping on X. The mapping T is called asymptotically regular at $x \in X$ if
$$\lim_{n\to\infty} M(T^n x, T^{n+1} x, t) = 1.$$

In [7] Golet introduced the concept of g-contraction in probabilistic metric space. Here we state the definition of g-contraction in the context of fuzzy metric space.

Definition 1.9 Let f and g be two mappings defined on fuzzy metric space (X, M, Δ) with values into itself. Then f is called fuzzy g-contraction if
$$t > 0 \text{ and } M(gx, gy, t) > 1 - t \text{ implies } M(fx, fy, kt) > 1 - kt, \text{ where } k \in (0,1).$$

Definition 1.10 Let f, g and T be three mappings defined on fuzzy metric space (X, M, Δ) with values into itself and let us take T is asymptotically regular at $x \in X$. Then f is called fuzzy asymptotically g-contraction with respect to T if
$$t > 0 \text{ and } M(gT^n x, gT^{n+1}x, t) > 1 - t \text{ implies } M(fT^n x, fT^{n+1}x, kt) > 1 - kt,$$
where $k \in (0,1)$.

Proposition 1.11 [7] Let g be a bijective mapping on (X, M, Δ) and f is a g-contraction on the fuzzy metric space (X, M, Δ) then $g^{-1}f$ is a continuous mapping on (X, M^g, Δ) with values into itself, where $M^g(x, y, t) = M(gx, gy, t)$.

Proof. Let $\{x_n\}_{n\geq 1}$ be a sequence in X such that $x_n \to x \in X$ under fuzzy metric M^g. This implies that $M(gx_n, gx, t) \to 1$ as $n \to \infty$, for every $t > 0$. By Definition 1.9, it follows that,

$$M(fx_n, fx, kt) \to 1 \text{ as } n \to \infty, \text{ for every } t > 0,$$

which implies, $M(gg^{-1}fx_n, gg^{-1}fx, kt) \to 1$ as $n \to \infty$, for every $t > 0$,

which implies, $M^g(g^{-1}fx_n, g^{-1}fx, kt) \to 1$ as $n \to \infty$, for every $t > 0$.

This shows that $g^{-1}f$ is a continuous mapping on (X, M^g, Δ) with values into itself.

2. Main Theorem

Theorem 2.1 Let (X, M,Δ) be a complete fuzzy metric space. Let f, g, T be three self mappings on X with g is bijective, f, T are g-contractive mappings and $fg^{-1}f = Tg^{-1}T$. Then f, g and T have a coincidence point, that is, there exists $p \in X$ such that fp = gp = Tp.

Proof. For t >1 and x, y $\in$ X, we have $M(gx, gy ,t) > 1 - t$. Now by the definition of g-contraction we can say,

$$M(fx, fy, kt) > 1 - kt, \text{ where } k \in (0,1).$$

Now, $M(fx, fy, kt) > 1 - kt$ implies,

$$M(gg^{-1}fx, gg^{-1}fy, kt) > 1 - kt,$$

that is, $M^g(g^{-1}fx, g^{-1}fy, kt) > 1 - kt$,

which implies, $M^g(hx, hy, kt) > 1 - kt$, where $g^{-1}f=h$.

Applying the same method we get after n-th iteration

$$M^g(h^n x, h^n y, k^n t) > 1 - k^n t.$$

For every $\varepsilon > 0$, $\lambda \in (0,1)$, there exist a positive $n(\varepsilon,\lambda)$ such that $k^n t \leq \min(\varepsilon, \lambda)$ for every $n \geq n(\varepsilon, \lambda)$.

Now, $M^g(h^n x, h^n y, \varepsilon) \geq M^g(h^n x, h^n y, k^n t) > 1 - k^n t > 1 - \lambda$.

Therefore, $M^g(h^n x, h^n y, \varepsilon) > 1 - \lambda$.

Let $x_0 \in X$ and let $\{x_n\}$ be the sequence of successive approximation defined as

$$x_{n+1} = hx_n.$$

Taking $x = x_{m-n}$ and $y = x_0$, from last inequality we get,

$$M^g(x_m, x_n, \varepsilon) > 1 - \lambda \text{ for every } n \geq n(\varepsilon, \lambda), \text{ and } m \geq n.$$

Therefore {xn} is Cauchy sequence in X.

Since, (X, M, Δ) is complete, then (X, M^g, Δ) is also complete and there exist a point $p \in X$ such that the sequence $\{x_n\}$ converges to p. Since, h is continuous on (X, M^g, Δ) we have, hp = p.

This implies that $g^{-1}fp = p$, that is, $fp = gp$.
Hence p is a coincidence point.
Again, $t > 0$ and $M(gx, gy, t) > 1 - t$ implies
$M(Tx, Ty, kt) > 1 - kt$, where $k \in (0,1)$.
Similarly we can show that there exists $q \in X$ such that, $Tq = gq$.

We can prove $p = q$ by using $fg^{-1}f = Tg^{-1}T$,
therefore, $fp = Tp = gp$,
that is, f, g and T have a coincidence point.

Remark. If $f = T$ and the space is complete Menger space, then the above result will be Golet's result.

Theorem 2.2 Let f, g, T be three mappings defined on complete fuzzy metric space (X, M, Δ) with values into itself where g is bijective, T is asymptotically regular at $x \in X$ and f is fuzzy asymptotically g-contraction with respect to T with $fT = Tf$, $gT = Tg$ at $x \in X$ and $g^{-1}f\,T$ is continuous in X. Then $fTx = gx$.

Proof. Since f is fuzzy asymptotically g-contraction with respect to T, by Definition 1.10 we get for $t > 0$,

$$M(gT^nx, gT^{n+1}x, t) > 1 - t,$$

which implies, $M(fT^nx, fT^{n+1}x, kt) > 1 - kt$, where $t > 0$ and $k \in (0, 1)$,
which implies, $M(gg^{-1}fT^nx, gg^{-1}fT^{n+1}x, kt) > 1 - kt$,
which implies, $M^g(g^{-1}fTT^{n-1}x, g^{-1}fTT^nx, kt) > 1 - kt$,
which implies, $M^g(\alpha T^{n-1}x, \alpha T^nx, kt) > 1 - kt$, putting $g^{-1}fT = \alpha$,
which implies, $M(f\alpha T^{n-1}x, f\alpha T^nx, k^2t) > 1 - k^2t$,
which implies, $M(fg^{-1}fTT^{n-1}x, fg^{-1}fTT^nx, k^2t) > 1 - k^2t$,
which implies, $M(fg^{-1}fTTT^{n-2}x, fg^{-1}fTTT^{n-1}x, k^2t) > 1 - k^2t$,
which implies, $M(fTg^{-1}fTT^{n-2}x, fTg^{-1}fTT^{n-1}x, k^2t) > 1 - k^2t$,
which implies, $M(gg^{-1}fTg^{-1}fTT^{n-2}x, gg^{-1}fTg^{-1}fTT^{n-1}x, k^2t) > 1 - k^2t$,
which implies, $M^g(\alpha^2T^{n-2}x, \alpha^2T^{n-1}x, k^2t) > 1 - k^2t$.

Continuing this process, we get

$$M^g(\alpha^nx, \alpha^nTx, k^nt) > 1 - k^nt.$$

For every $\varepsilon > 0$, $\lambda \in (0,1)$, there exist a positive $n(\varepsilon, \lambda)$ such that $k^nt \leq \min(\varepsilon, \lambda)$ for every $n \geq n(\varepsilon, \lambda)$.

Therefore, $M^g(\alpha^nx, \alpha^nTx, \varepsilon) \geq M^g(\alpha^nx, \alpha^nTx, k^nt) > 1 - k^nt > 1 - \lambda$,
which implies, $M^g(\alpha^nTx, \alpha^nx, \varepsilon) > 1 - \lambda$.

Let $\{x_n\}$ be a sequence defined by $x_{n+1} = \alpha x_n$. If we take $Tx = x_{m-n}$ and $x = x_0$, then we get from the above inequality,

$$M^g(x_m, x_n, \varepsilon) > 1 - \lambda, \text{ for every } n \geq n(\varepsilon,\lambda) \text{ and } m \geq n.$$

So, $\{x_n\}$ is a Cauchy sequence. Since, (X, M, Δ) is complete, then (X, M^g, Δ) is also complete. So, the sequence $\{x_n\}$ converges to $x \in X$. Again, α is continuous on (X, M^g, Δ), it follows that $\alpha x = x$, which implies $g^{-1}fTx = x$,

that is, $fTx = gx$.

Example 2.3 Let (X, M, Δ) be a complete fuzzy metric space where $X = [0, \infty)$ with $\Delta(a, b) = ab$ and let $M(x, y, t) = [\exp(|x-y|/t)]^{-1}$, for all $x,y \in X$ and $t \in (0, \infty)$. Then (X, M, Δ) be a complete fuzzy metric space. Define $f, g, T : X \to X$ by $fx = gx = x^3/4$ and $Tx = x^4/8$, for all $x \in X$. This example supports the Theorem 2.2.

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