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Jessen type functionals and exponential convexity

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Abstract

In this paper, we introduce the extension of Jessen functional and investigate logarithmic and exponential convexity. We also give mean value theorems of Cauchy and Lagrange type. Several families of functions are also presented related to our main results.

Keywords: Jessen functional, exponential convexity, mean value theorems.
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1. Introduction and preliminaries

Let $E(\neq \emptyset)$ and L be a linear class of real-valued functions $h : E \to \mathbb{R}$ having the properties:

L_1: $h, k \in L \Rightarrow (\alpha h + \beta k) \in L$, for all $\alpha, \beta \in \mathbb{R}$;

L_2: $1 \in L$ that is if $h(l) = 1$ for $l \in E$, then $h \in L$.

We also consider positive linear functionals $A : L \to \mathbb{R}$ possessing the properties:

A_1: $A(\alpha h + \beta k) = \alpha A(h) + \beta A(k)$ for $h, k \in L, \alpha, \beta \in \mathbb{R}$;

A_2: $h \in L, h(l) \geqslant 0$ on $E \Rightarrow A(h) \geqslant 0$ (A is positive).

The mapping A is said to be normalized if

A_3: $A(1) = 1$.

By a weight function, we mean a mapping $\omega : E \times E \to \mathbb{R}^+$ such that

$$\begin{aligned} A\left(\omega(x, y)\right) &= 1 \text{ (for each } y \text{ in } E), \\ B\left(\omega(x, y)\right) &= 1 \text{ (for each } x \text{ in } E), \end{aligned} \tag{1.1}$$

where A and B satisfy the properties A_1, A_2 and A_3.

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Jessen in [12] (see also [20]) gave the generalization of Jensen's inequality for positive linear functionals: For a continuous convex function $\Psi : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ and let L satisfy properties L_1, L_2 on a nonempty set E. If A is a positive linear functional on L with $A(1) = 1$, then for all $h \in L$ such that $\Psi(h) \in L$ we have $A(h) \in I$ and

$$\Psi(A(h)) \leqslant A(\Psi(h)).$$

Jessen functional is the difference of above inequality, written as

$$\Upsilon_1(\Psi) = A(\Psi(h)) - \Psi(A(h)).$$

For other notable literature about Jensen's inequality and related results see [1–3, 8]. The converse of Jessen's inequality [5] (see also [20]) is stated as:

Theorem 1.1. *For a convex function Ψ on an interval $I = [\eta, \zeta]$ $(-\infty < \eta < \zeta < \infty)$ and let L satisfy properties L_1, L_2 on a nonempty set E. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $h \in L$ such that $\Psi(h) \in L$ (so that $\eta \leqslant h(l) \leqslant \zeta$ for all $l \in E$), we have*

$$A(\Psi(h)) \leqslant \frac{\zeta - A(h)}{\zeta - \eta} \cdot \Psi(\eta) + \frac{A(h) - \eta}{\zeta - \eta} \cdot \Psi(\zeta).$$

In this paper we consider the functional defined from above inequality and is given as:

$$\Upsilon_2(\Psi) = \frac{\zeta - A(h)}{\zeta - \eta} \cdot \Psi(\eta) + \frac{A(h) - \eta}{\zeta - \eta} \cdot \Psi(\zeta) - A(\Psi(h)).$$

The following theorem is the refinement of Jessen's inequality.

Theorem 1.2. *Let L satisfy properties L_1 and L_2 on a nonempty set E, and assume that Ψ is a continuous convex function on an interval $I \subseteq \mathbb{R}$. If A and B are positive linear functionals with $A(1) = B(1) = 1$ and ω is a weight function (defined in (1.1)) then for all $h, h \cdot \omega \in L$ such that $\Psi(A(h \cdot \omega)), \Psi(h) \in L$ we have $A(h \cdot \omega), A(h) \in I$ and*

$$\Psi(A(h)) \leqslant B(\Psi(A(h \cdot \omega))) \leqslant A(\Psi(h)). \tag{1.2}$$

m-exponential convexity was firstly introduced by Pečarić and Perić in [19]. In [4, 7, 9, 10, 13–15, 18], the construction of m-exponentially convex functions is made through the method prescribed in [11]. The reader may refer to [6, 16, 17, 21] for the background of exponential convexity and mean value theorems.

In the next section, we prove the counterpart of the refined Jessen inequality and give its new version. In Section 3, we discuss m-exponential convexity of the functions associated with the linear functionals. We present different families of functions to investigate exponential convexity and log-convexity. Mean value theorems are also given in the last section.

Throughout paper, we assume ω to be a weight function which satisfies (1.1).

2. Main results

Now we prove the counterpart of the inequality $\Psi(A(h)) \leqslant B(\Psi(A(h \cdot \omega)))$ for compact interval $I = [\eta, \zeta]$.

Theorem 2.1. *Let Ψ be a convex function on $I = [\eta, \zeta]$ $(-\infty < \eta < \zeta < \infty)$. Let L satisfy properties L_1, L_2 on a nonempty set E, ω is weight function and A, B are isotonic linear normalized functionals on L, then for all $h \cdot \omega \in L$ such that $\Psi(A(h \cdot \omega)) \in L$ (so that $\eta \leqslant A(h \cdot \omega) \leqslant \zeta$), we have*

$$B(\Psi(A(h \cdot \omega))) \leqslant \frac{\zeta - A(h \cdot \omega)}{\zeta - \eta} \Psi(\eta) + \frac{A(h \cdot \omega) - \eta}{\zeta - \eta} \Psi(\zeta). \tag{2.1}$$

Proof. From the definition of convex function

$$\Psi(b) \leqslant \frac{c-b}{c-a}\Psi(a) + \frac{b-a}{c-a}\Psi(c), \quad (a \leqslant b \leqslant c, a < c).$$

Now set $a = \eta$, $b = A(h \cdot \omega)$, $c = \zeta$ give

$$\Psi(A(h \cdot \omega)) \leqslant \frac{\zeta - A(h \cdot \omega)}{\zeta - \eta}\Psi(\eta) + \frac{A(h \cdot \omega) - \eta}{\zeta - \eta}\Psi(\zeta).$$

Since B is isotonic linear and normalized functional, (2.1) holds. □

The next theorems are our main results.

Theorem 2.2. *Let Ψ be a convex function on $I = [\eta, \zeta]$ $(-\infty < \eta < \zeta < \infty)$. Let L satisfy properties L_1, L_2 on a nonempty set E, ω is weight function and A, B are isotonic linear normalized functionals on L, then for all $h, h \cdot \omega \in L$ such that $\Psi(A(h \cdot \omega)), \Psi(\eta + \zeta - A(h \cdot \omega)) \in L$ (so that $\eta \leqslant A(h \cdot \omega) \leqslant \zeta$), we have*

$$\Psi(\eta + \zeta - A(h)) \leqslant \Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h \cdot \omega))).$$

Proof. Since Ψ is continuous and convex, the same is also true for the function $\Phi : [\eta, \zeta] \to \mathbb{R}$ defined by $\Phi(t) = \Psi(\eta + \zeta - t), t \in [\eta, \zeta]$. Then by the left hand side of the inequality of (1.2), we have

$$\psi(A(h)) \leqslant B(\psi(A(h \cdot \omega))).$$

That is,

$$\Psi(\eta + \zeta + A(h)) \leqslant B(\Psi(\eta + \zeta - A(h \cdot \omega))).$$

Applying Theorem 2.1, we obtain

$$\begin{aligned} B(\Psi(\eta + \zeta - A(h \cdot \omega))) &\leqslant \frac{\zeta - A(h \cdot \omega)}{\zeta - \eta}\Phi(\eta) + \frac{A(h \cdot \omega) - \eta}{\zeta - \eta}\Phi(\zeta) \\ &\leqslant \frac{\zeta - A(h \cdot \omega)}{\zeta - \eta}\Psi(\zeta) + \frac{A(h \cdot \omega) - \eta}{\zeta - \eta}\Psi(\eta) \\ &= \Psi(\eta) + \Psi(\zeta) - \left[\frac{\zeta - A(h \cdot \omega)}{\zeta - \eta}\Psi(\eta) + \frac{A(h \cdot \omega) - \eta}{\zeta - \eta}\Psi(\zeta)\right] \\ &\leqslant \Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h \cdot \omega))). \end{aligned}$$

The last statement follows from the fact that if Ψ is concave, then $-\Psi$ is convex and A, B are linear on L. □

Theorem 2.3. *Let Ψ be a convex function on $I = [\eta, \zeta]$ $(-\infty < \eta < \zeta < \infty)$. Let L satisfy properties L_1, L_2 on a nonempty set E, ω is weight function and A, B are isotonic linear normalised functionals on L, then for all $h \cdot \omega \in L$ such that $\Psi(h), \Psi(\eta + \zeta - A(h \cdot \omega)) \in L$ (so that $\eta \leqslant h(l) \leqslant \zeta$ for all $l \in E$), we have*

$$B(\Psi(\eta + \zeta - A(h \cdot \omega))) \leqslant \Psi(\eta) + \Psi(\zeta) - A(\Psi(h)).$$

Proof. Analogous to the proof of Theorem 2.2 we can prove it by using right hand side of the inequality (1.2) and using Theorem 1.1 instead of Theorem 2.1. □

3. Exponential convexity

A real-valued function $k : I \to \mathbb{R}$ is m-exponentially convex if it is m-exponentially $\mathcal{J}$-convex and continuous on I. Hence it is an exponentially convex function (for detail see [19]).

Remark 3.1. A positive real-valued function $k : I \to \mathbb{R}$ is log-$\mathcal{J}$-convex if and only if it is 2-exponentially $\mathcal{J}$-convex. Converse is true provided that k is also continuous.

Remark 3.2. If the divided difference $[t_1, t_2; k] \geqslant 0$ for every $t_1, t_2 \in I$, then k is increasing on its domain.

Lemma 3.3 ([10]). *If $\Psi : I \to \mathbb{R}$ is log-convex, then for $l < r < s$ $(l, r, s \in I)$,*

$$(\Psi(r))^{s-l} \leqslant (\Psi(l))^{s-r} (\Psi(s))^{r-l}.$$

Remark 3.4. We consider the following functionals under the assumption of Theorems 2.2 and 2.3, respectively.

$$\Omega(\Psi) = \Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h \cdot \omega))) - \Psi(\eta + \zeta - A(h)), \tag{3.1}$$

$$\Omega'(\Psi) = \Psi(\eta) + \Psi(\zeta) - A(\Psi(h)) - B(\Psi(\eta + \zeta - A(h \cdot \omega))). \tag{3.2}$$

Then $\Omega(\Psi)$ and $\Omega'(\Psi)$ are positive.

We construct m-exponentially convex functions and exponentially convex functions by applying an elegant method from [11]. The following theorem produces new m-exponentially convex functions.

Theorem 3.5. *Let $J \subset \mathbb{R}$ be an open interval, and $\Theta = \{g_l | l \in J\}$ is a family of functions defined on $I \subset \mathbb{R}$, such that the function $l \mapsto [t_1, t_2, t_3; g_l]$ is m-exponentially $\mathcal{J}$-convex on J for all three different points $t_1, t_2, t_3 \in I$. Consider $\Omega(\Psi)$ as given in Remark 3.4. Then $l \mapsto \Omega(g_l)$ is m-exponentially $\mathcal{J}$-convex on J. If the function $l \mapsto \Omega(g_l)$ is continuous, then it is m-exponentially convex on J.*

Proof. Let $l_i, l_j \in J$, $l_{ij} = \frac{l_i + l_j}{2}$ and $a_i, a_j \in \mathbb{R}$ for $i, j \in \{1, 2, \cdots, m\}$ $(m \in \mathbb{N})$ and define the function Δ on I by

$$\Delta(t) = \sum_{i,j=1}^{m} a_i a_j g_{l_{ij}}(t).$$

Then Δ being the linear combination of continuous functions, is a continuous function. By assumption the function $l \mapsto [t_1, t_2, t_3; g_l]$ is m-exponentially $\mathcal{J}$-convex, therefore we have

$$[t_1, t_2, t_3; \Delta] = \sum_{i,j=1}^{m} a_i a_j [t_1, t_2, t_3; g_{l_{ij}}] \geqslant 0,$$

which implies that Δ is a convex function on I. Therefore we have $\Omega(\Delta) \geqslant 0$, which yields by the linearity of Ω, that

$$\sum_{i,j=1}^{m} a_i a_j \Omega(g_{l_{ij}}) \geqslant 0.$$

We conclude that the function $l \mapsto \Omega(g_l)$ is m-exponentially $\mathcal{J}$-convex function on J. □

The following corollaries are consequence of above theorem.

Corollary 3.6. *Let $J \subset \mathbb{R}$ be an open interval, and $\Theta = \{g_l | l \in J\}$ is a family of functions defined on $I \subset \mathbb{R}$, such that the function $l \mapsto [t_1, t_2, t_3; g_l]$ is exponentially $\mathcal{J}$-convex on J for all three different points $t_1, t_2, t_3 \in I$. Consider $\Omega(\Psi)$ as given in Remark 3.4. Then $l \mapsto \Omega(g_l)$ is an exponentially $\mathcal{J}$-convex function on J. If the function $l \mapsto \Omega(g_l)$ is continuous, then it is exponentially convex on J.*

Corollary 3.7. *Let $J \subset \mathbb{R}$ be an open interval, and $\Theta = \{g_l | l \in J\}$ is a family of functions defined on $I \subset \mathbb{R}$, such that the function $l \mapsto [t_1, t_2, t_3; g_l]$ is 2-exponentially $\mathcal{J}$-convex on J for all three different points $t_1, t_2, t_3 \in I$. Consider $\Omega(\Psi)$ as given in Remark 3.4. Then $l \mapsto \Omega(g_l)$ is 2-exponentially $\mathcal{J}$-convex function on J. If the function $l \mapsto \Omega(g_l)$ is continuous, then it is 2-exponentially convex on J, and thus log-convex, that is,*

$$\Omega^{l-r}(g_s) \leqslant \Omega^{l-s}(g_r)\Omega^{s-r}(g_l)$$

for $l, r, s \in J$ such that $r < s < l$.

Proof. This is an immediate consequence of Theorem 3.5 and Remark 3.1. □

Now we present different families of functions to investigate exponential convexity. The following lemma will be useful to construct new exponentially convex functions. Since the below mentioned result is the simple consequence of some basic examples and remarks given in [11], so we omit the proof.

Lemma 3.8.

(i) *For* $l > 0$, *let* $f_l : I = \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_l(t) = \frac{1}{l^2}\exp(lt).$$

Then $l \mapsto \frac{d^2}{dt^2} f_l(t)$ *is exponentially convex on* $(0, \infty)$ *for each* $t \in I$.

(ii) *For* $l > 1$, *let* $g_l : I = \mathbb{R}^+ \to \mathbb{R}$ *be defined by*

$$g_l(t) = \frac{t^l}{l(l-1)}.$$

Then $l \mapsto \frac{d^2}{dt^2} g_l(t)$ *is exponentially convex on* $(1, \infty)$ *for each* $t \in I$.

(iii) *For* $l > 1$, *let* $h_l : I = \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$h_l(t) = \frac{l^{-t}}{(\log l)^2}.$$

Then $l \mapsto \frac{d^2}{dt^2} h_l(t)$ *is exponentially convex on* $(1, \infty)$ *for each* $t \in I$.

(iv) *For* $l > 0$, *let* $k_l : I = \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$k_l(t) = \frac{1}{l}\exp(-t\sqrt{l}).$$

Then $l \mapsto \frac{d^2}{dt^2} k_l(t)$ *is exponentially convex on* $(0, \infty)$ *for each* $t \in I$.

To define the basic inequality of log-convex functions we present positive functionals.

Remark 3.9. The following positive functionals are useful in defining the basic inequality of log-convex functions.

$$\Lambda(f_l) = \frac{1}{l^2}\left(\exp(l\eta) + \exp(l\zeta) - B\left(\exp\left(lA(h\cdot\omega)\right)\right) - \exp(l\eta + l\zeta - lA(h))\right),$$

$$\Lambda(g_l) = \frac{1}{l(l-1)}\left(\eta^l + \zeta^l - B\left((A(h\cdot\omega))^l\right) - (\eta + \zeta - A(h))^l\right),$$

$$\Lambda(h_l) = \frac{1}{(\log l)^2}\left(l^{-\eta} + l^{-\zeta} - B\left(l^{-A(h\cdot\omega)}\right) - l^{A(h)-\eta-\zeta}\right),$$

$$\Lambda(k_l) = \frac{1}{l}\left(\exp(-\eta\sqrt{l}) + \exp(-\zeta\sqrt{l}) - B\left(\exp\left(-A(h\cdot\omega)\sqrt{l}\right)\right) - \exp\left((A(h) - \eta - \zeta)\sqrt{l}\right)\right).$$

Theorem 3.10. *Let* $\Omega(\Psi)$ *be the linear functional defined by* (3.1) *and define* $\phi_i : (0, \infty) \to \mathbb{R}$ *for* $i = 1, 4$ *and* $\phi_i : (1, \infty) \to \mathbb{R}$ *for* $i = 2, 3$ *by*

$$\phi_1(l) = \Lambda(f_l), \quad \phi_2(l) = \Lambda(g_l), \quad \phi_3(l) = \Lambda(h_l), \quad \phi_4(l) = \Lambda(k_l),$$

where f_l, g_l, h_l *and* k_l *are defined in Lemma* 3.8. *Then*

(i) *The function* ϕ_i *are continuous on* $(0, \infty)$ *for* $i = 1, 4$ *and continuous on* $(1, \infty)$ *for* $i = 2, 3$.

(ii) *If* $m \in \mathbb{N}$, $l_1, \cdots, l_n \in (0, \infty)$ *for* $i = 1, 4$ *and* $l_1, \cdots, l_n \in (1, \infty)$ *for* $i = 2, 3$, *then the matrices*

$$\left[\phi_i\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m},$$

are positive semidefinite.

(iii) *The functions* ϕ_i *are exponentially convex on* $(0, \infty)$ *for* $i = 1, 4$ *and exponentially convex on* $(1, \infty)$ *for* $i = 2, 3$.

(iv) *If* $l, r, s \in (0, \infty)$ *for* $i = 1, 4$ *and* $l, r, s \in (1, \infty)$ *for* $i = 2, 3$ *are such that* $l < r < s$, *then*

$$(\phi_i(r))^{s-l} \leqslant (\phi_i(l))^{s-r} (\phi_i(s))^{r-l},$$

where $\phi_i(l)$ *for* $i = 1, 2, 3, 4$ *are defined in Remark* 3.9.

Proof.
(i) The continuity of the functions $l \mapsto \phi_i(l)$ for $i \in \{1, 2, 3, 4\}$ is obvious.
(ii) Let $m \in \mathbb{N}$ and $d_j, l_j \in \mathbb{R}$ $(1 \leqslant j \leqslant m)$. Define the auxiliary function Δ_1 on $I = \mathbb{R}$ by

$$\Delta_1(t) = \sum_{j,k=1}^{m} d_j d_k f_{\frac{l_j + l_k}{2}}(t).$$

Since

$$\Delta_1''(t) = \sum_{j,k=1}^{m} d_j d_k \frac{d^2}{dt^2} f_{\frac{l_j + l_k}{2}}(t) \geqslant 0$$

for $t \in I$ by Lemma 3.8. This implies Δ_1 is convex. Now Theorem 2.2 implies that $\Omega(\Delta_1) \geqslant 0$. This means that

$$\left[\phi_1\left(\frac{l_j + l_k}{2}\right)\right]_{j,k=1}^{m}$$

is a positive semidefinite matrix.

To prove the remaining positive semidefinite matrices, we can define the auxiliary functions Δ_i for $i = 2, 3, 4$ in the similar manner.

(iii) and (iv) are simple consequences of (i), (ii) and Lemma 3.3. □

Remark 3.11. We can construct similar results for the positive functional $\Omega'(\Psi)$ defined in (3.2).

4. Mean value theorems

Now, we state the mean value theorems of Lagrange and Cauchy type.

The following lemma will be very useful.

Lemma 4.1 ([20]). *Let* $\Psi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, *be such that* $\Psi \in C^2(I)$, Ψ'' *is bounded and* $m = \inf_{l \in I} \Psi''(l)$, $M = \sup_{l \in I} \Psi''(l)$. *Then the functions* $\Psi_1, \Psi_2 : I \to \mathbb{R}$ *defined by*

$$\Psi_1(l) = \frac{M}{2} l^2 - \Psi(l),$$
$$\Psi_2(l) = \Psi(l) - \frac{m}{2} l^2,$$

are convex.

Theorem 4.2. *Let* L *satisfy properties* L_1 *and* L_2 *on a nonempty set* E*, and let* $\Psi : I \to \mathbb{R}$, $\Psi \in C^2(I)$, *where* $I = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. *If* A, B *are isotonic linear normalised functionals and* ω *is a weight function (defined in* (1.1)*), then for all* $h, h\cdot\omega \in L$ *such that* $\Psi(A(h\cdot\omega)), (A(h\cdot\omega))^2 \in L$ *there exists some* $\gamma \in I$ *such that the following holds*

$$\Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h\cdot\omega))) - \Psi(\eta+\zeta-A(h)) = \alpha\Psi''(\gamma), \tag{4.1}$$

where

$$\alpha = \frac{1}{2}\left[\eta^2 + \zeta^2 - (\eta+\zeta-A(h))^2 - B([A(h\cdot\omega)]^2)\right].$$

Proof. Denote $M = \max_{l\in I}\Psi''(l)$ and $m = \min_{l\in I}\Psi''(l)$. Then by Lemma 4.1, the functions $\Psi_1, \Psi_2 : I \to \mathbb{R}$ are convex. Since they are also continuous. Applying Theorem 2.2, we get

$$\Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h\cdot\omega))) - \Psi(\eta+\zeta-A(h)) \leqslant \alpha M,$$

and

$$\Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h\cdot\omega))) - \Psi(\eta+\zeta-A(h)) \geqslant \alpha m.$$

Now combining these two inequalities and since Ψ'' is continuous, there exists $\gamma \in I\,(m \leqslant \Psi''(\gamma) \leqslant M)$ such that (4.1) holds. □

Theorem 4.3. *Let* L *satisfy properties* L_1 *and* L_2 *on a nonempty set* E*, and let* $\Phi, \Psi : I \to \mathbb{R}, \Phi, \Psi \in C^2(I)$, *where* $I = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. *If* A, B *are isotonic linear normalised functionals and* ω *is a weight function (defined in* (1.1)*), then for all* $h, h\cdot\omega \in L$ *such that* $\Phi(A(h\cdot\omega)), \Psi(A(h\cdot\omega)), (A(h\cdot\omega))^2 \in L$ *and* $\eta^2 + \zeta^2 - (\eta+\zeta-A(h))^2 - B([A(h\cdot\omega)]^2) \neq 0$ *there exists some* $\gamma \in I$ *such that the following holds*

$$\begin{aligned}\Psi''(\gamma)&\left[\Phi(\eta) + \Phi(\zeta) - B(\Phi(A(h\cdot\omega))) - \Phi(\eta+\zeta-A(h))\right]\\ &= \Phi''(\gamma)\left[\Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h\cdot\omega))) - \Psi(\eta+\zeta-A(h))\right].\end{aligned}$$

Proof. Consider the function $k \in C^2(I)$ defined as $k = c_1\Phi - c_2\Psi$, where c_1 and c_2 are defined by

$$c_1 = \Psi(\eta) + \Psi(\zeta) - B(\Psi(A(h\cdot\omega))) - \Psi(\eta+\zeta-A(h)),$$

and

$$c_2 = \Phi(\eta) + \Phi(\zeta) - B(\Phi(A(h\cdot\omega))) - \Phi(\eta+\zeta-A(h)).$$

Since $k \in C^2(I)$, now by applying Theorem 4.2 on the function k, it follows that there exists some $\gamma \in I$ such that the following holds

$$k(\eta) + k(\zeta) - B(k(A(h\cdot\omega))) - k(\eta+\zeta-A(h)) = \alpha k''(\gamma).$$

The left-hand side of this equation equals to zero, since $\alpha \neq 0$, so we have that $k''(\gamma) = 0$. Thus the assertion of our theorem follows directly. □

Similarly we can define mean value theorems for Theorem 2.3. Here we omit the proofs.

Theorem 4.4. *Let* L *satisfy properties* L_1 *and* L_2 *on a nonempty set* E*, and let* $\Psi : I \to \mathbb{R}, \Psi \in C^2(I)$, *where* $I = [\eta, \zeta] \subseteq \mathbb{R}\,(-\infty < \eta < \zeta < \infty)$. *If* A, B *are isotonic linear normalised functionals and* ω *is a weight function (defined in* (1.1)*), then for all* $h\cdot\omega \in L$ *such that* $\Psi(\eta+\zeta-A(h\cdot\omega))$, $\Psi(h)$, $(\eta+\zeta-A(h\cdot\omega))^2$, $h^2 \in L$ *there exists some* $\gamma \in I$ *such that the following holds*

$$\Psi(\eta) + \Psi(\zeta) - A(\Psi(h)) - B(\Psi(\eta+\zeta-A(h\cdot\omega))) = \beta\Psi''(\gamma),$$

where

$$\beta = \frac{1}{2}\left[\eta^2 + \zeta^2 - A(h^2) - B([\eta+\zeta-A(h\cdot\omega)]^2)\right].$$

Theorem 4.5. *Let* L *satisfy properties* L_1 *and* L_2 *on a nonempty set* E, *and let* $\Phi, \Psi : I \to \mathbb{R}$, $\Phi, \Psi \in C^2(I)$, *where* $I = [\eta, \zeta] \subseteq \mathbb{R}$ $(-\infty < \eta < \zeta < \infty)$. *If* A, B *are isotonic linear normalised functionals and* ω *is a weight function (defined in* (1.1)*), then for all* $h \cdot \omega \in L$ *such that* $\Phi(\eta + \zeta - A(h \cdot \omega))$, $\Psi(\eta + \zeta - A(h \cdot \omega))$, $\Phi(h)$, $\Psi(h)$, $(\eta + \zeta - A(h \cdot \omega))^2$, $h^2 \in L$ *and* $\eta^2 + \zeta^2 - A(h^2) - B([\eta + \zeta - A(h \cdot \omega)]^2) \neq 0$ *there exists some* $\gamma \in I$ *such that the following holds*

$$\begin{aligned}\Psi''(\gamma) &[\Phi(\eta) + \Phi(\zeta) - A(\Phi(h)) - B(\Phi(\eta + \zeta - A(h \cdot \omega)))] \\ &= \Phi''(\gamma) [\Psi(\eta) + \Psi(\zeta) - A(\Psi(h)) - B(\Psi(\eta + \zeta - A(h \cdot \omega)))] .\end{aligned}$$

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