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A Note on Generalization of Classical Jensen's Inequality

P.O. Olanipekun, A. Mogbademu

Departmentof Mathematics, Universityof Lagos, Yaba-Nigeria

peter.olanipekun1@students.unilag.edu.ng, amogbademu@unilag.edu.ng

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Abstract

In this note, we prove a new generalisation of the Jensen's inequality by using a Riemann-Stieltjes integrable function and convex functions under a mild condition. An example was given to support the claims of this paper.

Keywords: Convex functions, Jensen's inequality.

1. Introduction and Preliminaries

In [5], Royden and Fitzpatrick, examined the classical form of Jensen's inequality [3]

$$\varphi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 (\varphi o f)(x)dx. \qquad (1.1)$$

Using the notion of the supporting line that exists at the point $(\alpha, \varphi(\alpha))$ for the graph of φ *where* $\alpha \in (0, 1)$. Indeed, they gave a short proof for the Jensen's inequality. The purpose of this paper is to employ a simple analytic technique which is independent of the idea in [4] to show that for any two convex functions $\varphi(x), \beta(x)$ and another Riemann Stieltjes Integrable function *f*(x) defined on [a, b] then

$$\varphi\left(\int_a^b f d\beta\right) \leq \int_a^b \varphi(f) d\beta \qquad (1.2)$$

under a mild condition.

Remark: A case where $\beta(x)$ is the identity function and $b-a=1$gives the kind of Jensen's inequality discussed in [5].

The following well known definition and Lemmas are useful in the proof of our results.

Definition 1.1. A function φ is convex on [a, b] if,

$\varphi(x) \leq \varphi(y) + \frac{\varphi(t)-\varphi(y)}{t-y}(x-y)$,where$a \leq y \leq x \leq t \leq b$.

Lemma 1.1 ([1, 2][5]). Suppose φ is convex on [a, b] and differentiable at $\alpha \in (a,b)$, then,

$$\varphi(\alpha)+\varphi^{'}(\alpha)(x-\alpha) \leq \varphi(x),\ \forall\, x \in [\mathrm{a},\mathrm{b}]\ .$$

Proof: See Lemma 1 of [2] and Theorem 18 in Chapter 6 of [5].

Lemma 1.2 [3]. Let φ be an increasing function on the closed bounded interval [a, b], then $\varphi^{'}$ is integrable over [a, b] and $\int_a^b \varphi^{'} \leq \varphi(b)-\varphi(a)$.

Proof: See Corollary 4 in section 6. 2 of [5].

2. Main results

Theorem 2.1. *Let $\varphi(x)$, $\beta(x)$ be convex functions on $(-\infty,\infty)$ and $f(x)$ Riemann-Stieltjes integrable w.r.t $\beta(x)$ over [a, b] such that $\beta(b)-\beta(a)=1$. Then,*

$$\varphi\left(\int_a^b f(x)d\beta\right) \leq \int_a^b (\varphi o f)(x)d\beta.$$

Proof. Let $\alpha = \int_a^b f d\beta$. (2. 1)

Choose $m \in \mathbb{R} \ni y = m(t-\alpha)+\varphi(\alpha)$ is the equation of the supporting line passing through $(\alpha,\varphi(\alpha))$ for the graph of φ. Clearly, $\varphi^{'}(\alpha^{-}) < m < \varphi^{'}(\varphi^{+})$. From Lemma 1. 1, we have:

$$\varphi(t) \geq m(t-\alpha)+\varphi(\alpha)\ \forall\, t \in \mathbb{R}. \quad (2.\ 2)$$

And, in particular

$$\varphi\big(f(x)\big) \geq m[f(x)-\alpha]+\varphi(\alpha)\ for\ x \in [a,b]\ (2.\ 3)$$

Integrating both sides of (2.3)

$$\int_a^b \varphi\big(f(x)\big)d\beta \geq \int_a^b (m[f(x)-\alpha]+\varphi(\alpha))d\,\beta$$

$$= m\int_a^b f(x)d\beta - m\alpha[\beta(b)-\beta(a)]+\varphi(\alpha)[\beta(b)-\beta(a)]$$

$$= m\alpha - m\alpha + \varphi(\alpha)$$

$$= \varphi(\int_a^b f d\beta) . \quad (2.4)$$

That is, $\int_a^b (\varphi o f) d\beta \geq \varphi\left(\int_a^b f(x) d\,\beta\right)$ completing the Proof.

Example

Let $\beta(x) = \begin{cases} 0, & x = a \\ 1, & a < x \leq b \end{cases}$

Clearly, $\beta(b) - \beta(a) = 1$ and for any convex function φ and Riemann Integrable function f on [a, b], then Theorem 2.1 holds.

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